

ЗАХТЕВ ЗА ОДОБРАВАЊЕ ТЕМЕ ДОКТОРСКЕ ДИСЕРТАЦИЈЕ

ОПШТИ И КОНТАКТ ПОДАЦИ КАНДИДАТА

Презиме, име једног родитеља и име	Tasić Radosav Marija Тасић Радосав Марија
Матични број	2209988747037
Датум и место рођења	22.09.1988. Vranje 22.09.1988. Врање
Адреса сталног боравка	Kozjačka 6/7, Vranje Козјачка 6/7, Врање
Број фиксног и мобилног телефона	0693524774
Е-пошта	marijajordanovic@hotmail.com, marija.jordanovic@pmf.edu.rs
Назив студијског програма (за студенте докторских студија)	PhD School of Mathematics Докторска школа математике

УНИВЕРЗИТЕТ У НИШУ
ПРИРОДНО-МАТЕМАТИЧКИ ФАКУЛТЕТ У НИШУ

Примљено: 20.4.2026.

Орг. јед.	Број	Прилог	Вредност
01	745		

БИОГРАФИЈА КАНДИДАТА

ОБРАЗОВАЊЕ

Основне студије

Универзитет	University of Belgrade / Универзитет у Београду
Факултет	Faculty of Mathematics / Математички факултет
Студијски програм	Professor of Mathematics and Computer Science / Професор математике и рачунарства
Звање	Bachelor of Mathematics / Дипломирани математичар
Година уписа	2007
Година завршетка	2011
Просечна оцена	8.05 (осам, 5/100)

Магистарске студије

Универзитет	University of Belgrade / Универзитет у Београду
Факултет	Faculty of Mathematics / Математички факултет
Студијски програм	Professor of Mathematics and Computer Science / Професор математике и рачунарства
Звање	Master of Mathematics / Мастер математичар
Година уписа	2011
Година завршетка	2012
Просечна оцена	10.00 (десет)
Научна област	Mathematics / Математика
Наслов завршног рада	An e-course on functions in programming Електронски курс о функцијама у програмирању

Остало

(Чланство у научним удружењима, награде, студијски боровци у иностранству, учешће у пројектима и др.)

Membership in professional associations:

- Member of the State Commission for Informatics Competitions for Primary School Students (2021-).
- Team Leader of the Serbian team at the Junior Balkan Olympiad in Informatics (JBOI), Dorohoi, Romania (2022).
- State seminar on teaching mathematics and computer science, workshop lecturer (2023).
- Member of the Commission for State Informatics Competition (2023) and Regional Informatics Competitions (2024-2026), Regional Center for Talents Vranje.
- Member of the Promotion Committee, Pedagogical Faculty in Vranje (2025).

Чланство у стручним удружењима

- Члан Државне комисија за такмичења ученика основних школа из информатике (2021-).
- Руководилац тима Републике Србије на Јуниорској информатичкој балканијади (JBOI), Дорохој, Румунија (2022).

- Државни семинар Друштва математичара Србије, предавач на радионицама (2023).
- Члан Комисије за државно такмичење из информатике (2023) и регионално такмичење из информатике (2024-2026), Регионални центар за таленте у Врању.
- Члан Комисије за промоцију Педагошког факултета Врање (2025/2026).

Радно искуство

Занимање	Teaching Associate / Сарадник у настави (2012-2013) Teaching Assistant / Асистент (2013-)
Радна организација у којој је запослен, место и адреса	University of Niš, Pedagogical Faculty in Vranje, Partizanska 14, 17500 Vranje Универзитет у Нишу, Педагошки факултет у Врању, Партизанска 14, 17500 Врање

БИБЛИОГРАФИЈА НАУЧНИХ И СТРУЧНИХ РАДОВА

(радове рангирати по важности од оних са највећом категоријом)

Р. бр.	Аутор-и, наслов, часопис, година, број волумена, странице	Категорија
1.	M. Tasić , M. Ćirić, J. Ignjatović, S. Stanimirović, M. Stanković, <i>Algebraic characterization of hesitant fuzzy relation inequalities and equations over power set quantales</i> , FILOMAT (2026), accepted for publication.	M21
2.	M. Stanković, M. Tasić , Đ. Stakić, A. Damjanović, <i>Exploring Wikipedia Usage in Primary School Education: Students' Experiences and Perspectives</i> , International Journal of Cognitive Research in Science, Engineering and Education, vol. 13, no. 2, 2025, pp. 323-334, doi: 10.23947/2334-8496-2025-13-2-323-334	M22
3.	Đ. Stakić, M. Tasić, M. Stanković , and M. Bogdanović, <i>Students' Attitudes Towards the Use of Wikipedia: A Teaching Tool and a Way to Modernize Teaching</i> , Área Abierta, vol. 21, no. 2, 2021, pp. 309-25, doi: https://doi.org/10.5209/arab.72760 .	M23
3.	S. Anđelković, M. Tasić , A. Milanović, Educational Computer Software Aimed at Improving the Quality of Elementary Mathematics Instruction. <i>1st International Scientific Conference Teaching, Learning and Teacher Education (TLTE), Quality of Teaching and Learning (QTL) - Reality and Possibilities</i> , pp. 349-364, Pedagogical Faculty in Vranje, 2023.	M33
4.	M. Tasić , Креирање електронских курсева помоћу Canvas система за управљање учењем. <i>Имплементација нових технологија у образовни систем, зборник радова</i> , стр. 85-96, Педагошки факултет, Врање, 2022.	M63
5.	A. Анђелковић, M. Tasić, Дигитални рефлексивни алати у универзитетској настави и учењу. <i>XVIII скуп Трендови развоја: универзитетско образовање за привреду</i> (216-218). Копаоник: Универзитет у Новом Саду, Факултет техничких наука - Нови Сад, 2022.	M63
7.	M. Богдановић, M. Tasić , <i>Практикум из информатике</i> , Врање: Педагошки факултет у Врању, 2021, ISBN 978-86-6301-043-7 COBISS.SR-ID 51784969.	
8.	M. Богдановић, M. Станковић, M. Tasić , <i>Елементарни математички појмови – теорија и задаци</i> , Врање: Педагошки факултет у Врању, 2019, ISBN 978-86-6301-031-4, COBISS.SR-ID 279809804.	
9.	M. Богдановић, M. Станковић, M. Јордановић , <i>Some comments on isometries of sphere</i> . Годишњак Педагошког факултета у Врању, књига VIII, 1/2017, стр. 65-73, 2017.	M52
10.	C. Јанковић, M. Јордановић , <i>Компјутерске игре у функцији развијања почетних математичких појмова</i> , Шеста међународна Конференција ТИО, Зборник радова ТИО 2016, стр. 161-165, Чачак, 28-29. мај 2016; Универзитет у Крагујевцу, Факултет техничких наука, Чачак 2016.	M33
11.	M. Станковић, Ђ. Стакић, M. Богдановић, M. Јордановић , <i>Википедија као могућност за иновације у настави Информатике на примеру Учитељског факултета</i> , Иновације у настави XXVIII, 2015/4, стр. 114-120, 2015.	M52
12.	M. Јордановић , <i>Предности Сингапурске математике</i> , Могућности модификовања методике васпитно-образовног рада и методике разредне наставе, тематски зборник, стр. 161-173, Врање 2014.	M45
13.	M. Станковић, M. Јордановић , C. Јанковић, <i>Примена програмског пакета GeoGebra у циљу осавремењивања наставе математике</i> , Компетенције васпитача за друштво знања, тематски зборник, стр. 547-554, Кикинда 25. октобар 2014.	M53
14.	M. Јордановић , Н. Миленковић, <i>Решавање проблема трговачког путника методом симулираног каљења</i> , Годишњак Учитељског факултета у Врању, Учитељски факултет у Врању, стр. 51-61, Врање 2014.	M52

15.	М. Јордановић, М. Станковић, Moodle платформа као софтверска подршка савременој настави, Савремене тенденције у наставним и ваннаставним активностима на учитељским (педагошким) факултетима, тематски зборник, стр. 364-376, Врање 2014.			M45
ПРЕДЛОГ ТЕМЕ ДОКТОРСKE ДИСЕРТАЦИЈЕ				
Предлог наслова теме докторске дисертације	Неодлучне фази релацијске неједначине и једначине над кванталом подскупова полугрупе (превод главног наслова)			
Предлог наслова теме на енглеском језику	Hesitant fuzzy relation inequalities and equations over the power set quantale of a semigroup (главни наслов)			
Научно поље	Natural Sciences and Mathematics / Природно-математичке науке			
Научна област	Mathematical Sciences / Математичке науке			
Ужа научна област	Mathematical Sciences / Математичке науке			
Научна дисциплина	Algebra and Mathematical Logic / Алгебра и математичка логика			
ПРЕДЛОГ МЕНТОРА				
Име и презиме	Miroslav Ćirić Мирослав Ћирић			
Звање	Full professor Редовни професор			
Научна и ужа научна област за коју је изабран у звање	Mathematical Sciences Математичке науке			
Датум избора	8.5.2000.			
Установа у којој је запослен	University of Niš, Faculty of Sciences and Mathematics, Višegradska 33, P.O. Box 224, 18000 Niš Универзитет у Нишу, Природно-математички факултет у Нишу, Вишеградска 33, Пош. фак: 224, 18000 Ниш			
Е-пошта	miroslav.ciric@pmf.edu.rs			
Пет најзначајнијих радова ментора у последњих десет година из научне области којој припада тема докторске дисертације				
Р. бр.	Аутор-и, наслов, часопис, година, број волумена, странице			Категорија
1.	S. Stanimirović, L. A. Nguyen, M. Ćirić, M. Stanković, Approximate state reduction of fuzzy finite automata, FUZZY SETS AND SYSTEMS 519 (2025) 109535.			M21a+
2.	M. Stanković, M. Ćirić, S. Stanimirović, Đ. Stakić, Approximate bisimulations for Kripke models of fuzzy multimodal logics over complete Heyting algebras, FUZZY SETS AND SYSTEMS 507 (2025) 109299.			M21a+
3.	S. Stanimirović, L. A. Nguyen, M. Ćirić, M. Stanković, Breadth-first fuzzy bisimulations for fuzzy automata, FUZZY SETS AND SYSTEMS 503 (2025) 109246.			M21a+
4.	I. Micić, S. Stanimirović, J. R. González de Mendivil, M. Ćirić, Z. Jančić, Finite determinization of fuzzy automata using a parametric product-based t-norm, FUZZY SETS AND SYSTEMS 488 (2024) 108990.			M21a+
5.	I. Micić, M. Ćirić, J. Matejić, S. Stanimirović, L. A. Nguyen, Approximate weak simulations and bisimulations for fuzzy automata over the product structure, FUZZY SETS AND SYSTEMS 485 (2024) 108959.			M21a+
Менторства у последње три године				
Р. бр.	Име и презиме докторанда, тема докторске дисертације, факултет/универзитет		Датум именов.	Датум одбране
1.	Marko Stanković, Bisimulations for Kripke models of Fuzzy Multimodal Logics, University of Niš, Faculty of Sciences and Mathematics, 2023 (Study Program: PhD School of Mathematics) Марко Станковић, Бисимулације за Крипкеове моделе фази мултимодалних логика, Универзитет у Нишу, Природно-математички факултет, 2023 (студијски програм: Докторска школа математике).			6.2.2023.
ОБРАЗЛОЖЕЊЕ ТЕМЕ				
1. Subject of scientific research (до 800 речи) Fuzzy set theory was introduced by Lotfi A. Zadeh, as a mathematical framework for modeling uncertainty, vagueness, and imprecision. Building on this foundational work, fuzzy set theory has evolved into a complex scientific discipline with numerous extensions aimed at modeling different types of uncertainty more precisely. Classical fuzzy sets represent uncertainty by assigning each element a single membership degree from the interval $[0,1]$. Although this approach has proven effective in many applications, it can be limited in situations where it is				

not possible to determine a unique membership value due to incomplete information, cognitive limitations, or conflicting assessments.

To address this limitation, Torra introduced hesitant fuzzy sets (HFSs) as a natural generalization of fuzzy sets that explicitly accounts for hesitation in membership assessment. Within the HFS framework, the membership degree of an element is expressed as a set of possible values from the interval $[0,1]$, called a hesitant fuzzy element, allowing the simultaneous consideration of multiple alternative membership values. This approach is particularly well suited to modeling human judgments, where hesitation arises from uncertainty, lack of confidence, or disagreement among experts.

Hesitant fuzzy sets have been successfully applied in a wide range of domains, including decision support systems, supplier selection, risk assessment, medical diagnosis, pattern recognition, and social network analysis.

A central issue in the theory of hesitant fuzzy sets is the handling of hesitant information through appropriate operations on hesitant fuzzy elements. One of the foundational approaches to defining operations on HFSs is based on lifting classical fuzzy set operations to hesitant fuzzy elements. In this framework, binary operations such as union, intersection and complement, as well as algebraic sum and product, are defined by applying the corresponding fuzzy operation to every pair of values across the relevant hesitant fuzzy elements.

In addition, the algebraic aspect of HFS theory has also been developed. It has been shown that, under certain conditions, hesitant fuzzy sets admit a lattice structure and even residuated lattices. These algebraic structures provide a theoretical foundation for approximate reasoning and systematic decision-making under hesitation. However, they are defined only for specialized classes of hesitant fuzzy sets or for specific orderings, which indicates certain limitations in terms of applicability.

Although different types of lattice-valued fuzzy sets and relations have been studied in the literature, most existing approaches are formulated within classical residuated lattices or other fixed algebraic structures, where uncertainty is typically represented by a single membership degree. In contrast, hesitant information involves sets of possible values, which requires an algebraic framework that can naturally handle such multi-valued information while still supporting the study of relation inequalities and equations. This motivates the use of power set quantales as the underlying membership structure, since they provide a natural setting for extending fuzzy relational methods to the hesitant context, while preserving the residuation property.

The subject of this dissertation is the introduction and analysis of a new algebraic framework within HFS theory. The framework is based on the power set quantale of a semigroup and enables the extension of classical and fuzzy relational properties to the hesitant fuzzy context. One of the central aspects of the research is the characterization of hesitant fuzzy equivalence relations, their corresponding classes and partitions, as well as the characterization of quasi-orders, including the construction of an equivalence relation from a given quasi-order.

Further, the research includes the study of systems of hesitant fuzzy relation inequalities and equations, with one side containing the composition of an unknown hesitant fuzzy set and given hesitant fuzzy relations over a power set quantale, while the other side contains only the same unknown hesitant fuzzy set or another one. In particular, our research includes the characterization of solution sets of these systems and the development of algorithms for computing their greatest solutions. Moreover, the characterization of solution sets has been introduced, in terms of closure and opening systems in the quantale of hesitant fuzzy sets, together with the corresponding closure and opening operators. Finally, the subjects of our investigation have been more general non-linear systems of hesitant fuzzy relation inequalities and equations, called hesitant weakly linear systems. Two types of weakly linear systems will be analyzed within the proposed framework: homogeneous which include only fuzzy relations on a single set, and heterogeneous one composed of fuzzy relations on two possible different sets.

Предмет научног истраживања

Теорију фази скупова увео је Lotfi A. Zadeh, као математички оквир за моделовање неизвесности, неодређености и непреизности. Полазећи од овог фундаменталног рада, теорија фази скупова еволуирала је у комплексну научну дисциплину са бројним проширењима развијеним ради прецизнијег моделовања различитих видова неизвесности. Класични фази скупови моделују неизвесност тако што сваком елементу

придружују јединствен степен припадности из интервала $[0,1]$. Иако се овај приступ показао ефикасним у бројним применама, он може бити ограничен у ситуацијама у којима није могуће одредити јединствену вредност степена припадности услед непотпуних информација, когнитивних ограничења или противречних процена.

У циљу превазилажења овог ограничења, Тогта је увео неодлучне фази скупове (НФС) као природну генерализацију фази скупова, који експлицитно моделују оклевање при процени степена припадности. У оквиру теорије неодлучних фази скупова, степен припадности елемента представљен је скупом могућих вредности из интервала $[0,1]$, који се назива неодлучни фази елемент. Тиме се омогућава истовремено разматрање више алтернативних вредности степена припадности. Овај приступ посебно је погодан за моделовање људских процена, у којима се оклевање јавља услед неизвесности, недостатка самопоуздања или несагласности међу експертима.

Неодлучни фази скупови нашли су широку примену у различитим областима, укључујући системе за подршку одлучивању, избор добављача, процену ризика, медицинску дијагностику, препознавање образаца и анализу друштвених мрежа.

Централно питање у теорији неодлучних фази скупова односи се на обраду информација са оклевањем применом одговарајућих операција над неодлучним фази елементима. Један од основних приступа дефинисању таквих операција заснива се на подизању (lifting) класичних операција дефинисаних над фази скуповима на ниво неодлучног фази елемента. У том контексту, бинарне операције као што су унија, пресек и комплемент, као и алгебарски збир и производ, дефинишу се применом одговарајућих фази операција на све парове вредности из датих неодлучних фази елемента.

Поред наведеног, развијен је и алгебарски аспект теорије неодлучних фази скупова. Показано је да неодлучни фази скупови под одређеним условима формирају структуру мреже, па чак и резидуиране мреже. Ове алгебарске структуре пружају теоријску основу за апроксимативно закључивање и систематско доношење одлука у условима оклевања. Међутим, оне су дефинисане само за специјализоване класе неодлучних фази скупова или за специфична уређења, што указује на извесна ограничења у погледу примене.

Иако су у литератури проучавани различити типови фази скупова и релација са вредностима у мрежама, већина постојећих приступа формулисана је у оквиру класичних резидуираних мрежа или других фиксних алгебарских структура, при чему је неизвесност обично представљена једним степеном припадности. Насупрот томе, информације са оклевањем подразумевају скупове могућих вредности, што природно захтева адекватан алгебарски оквир, који може да обради такве вишевредносне информације, док истовремено омогућава анализу релацијских неједначина и једначина. То нас је мотивисало да као основну структуру припадности разматрамо квантале подскупова полугрупе, будући да они представљају природан оквир за проширење фази релацијских метода на контекст неодлучних фази релација, уз очување својства резидуације.

Предмет истраживања ове дисертацији јесте увођење и анализа новог алгебарског оквира у теорији неодлучних фази скупова. Овај оквир заснива се на кванталу подскупова полугрупе и омогућава проширење класичних и фази релацијских својстава на контекст неодлучних фази релација. Један од централних аспеката истраживања јесте карактеризација неодлучних фази релација еквиваленције, њихових одговарајућих класа и партиција, као и карактеризација квази-уређења, укључујући и конструисање релације еквиваленције од датог квази-уређења.

Поред тога, истраживање обухвата проучавање система неједначина и једначина неодлучних фази релација, где једна страна садржи композицију непознатог неодлучног фази скупа и датих неодлучних фази релација над кванталом партитивног скупа, док друга страна садржи искључиво тај исти или неки други непознати неодлучни фази скуп. Специјално, наше истраживање укључује карактеризацију скупова решења ових система и развој алгоритама за израчунавање њихових највећих решења. Штавише, уведена је карактеризација скупова решења преко система затворења и отворења у кванталу неодлучних фази скупова, заједно са одговарајућим операторима затворења и отворења. Коначно, предмет нашег истраживања јесу и општији нелинеарни системи неједначина и једначина неодлучних фази релација, названи неодлучни слабо линеарни системи. У оквиру предложеног модела биће анализирана два типа слабо линеарних система: хомогени, који обухватају само неодлучне фази

релације на једном скупу, и хетерогени, састављени од неодлучних фази релација на два потенцијално различита скупа.

2. Aim of scientific research (до 500 речи)

Research in this area has several important objectives.

The main objective of this research is the introduction and analysis of a new algebraic framework for hesitant fuzzy sets, based on the power set quantale of a semigroup. Within the proposed framework, the objective is to develop a general theoretical setting that enables the extension of classical and fuzzy relational methods within the HFS theory, while preserving the residuation property.

In particular, the research will focus on:

- the characterization of hesitant fuzzy equivalence relations, including the corresponding equivalence classes and partitions of the set,
- the characterization of hesitant fuzzy quasi-orders,
- the construction of a hesitant fuzzy equivalence relation starting from a given hesitant fuzzy quasi-order,
- the study of systems of inequalities and equations with one or two unknown hesitant fuzzy sets,
- the study of systems of inequalities and equations with one or two unknown hesitant fuzzy relations,
- the development of algorithms for determining the greatest solutions of the aforementioned systems of inequalities and equations.

The focus of the research is to establish a theoretically consistent and computationally efficient algebraic framework that enables further analysis of hesitant fuzzy systems of relation inequalities and equations, as well as to develop concrete procedures for solving these systems within the proposed framework.

Циљ научног истраживања

Истраживање у овој области има неколико важних циљева.

Основни циљ овог истраживања јесте увођење и анализа новог алгебарског оквира за неодлучне фази скупове, заснованог на кванталу подскупова полугрупе. У оквиру предложеног оквира, циљ је развити општу теоријску поставку која омогућава проширење класичних и фази релацијских метода у оквиру теорије неодлучних фази скупова, уз очување својства резидуације.

Посебно, истраживање ће се усмерити на:

- карактеризацију неодлучних фази релација еквиваленције, укључујући одговарајуће класе еквиваленције и партиције скупа,
- карактеризацију неодлучних фази квази-уређења,
- конструкцију неодлучне фази релације еквиваленције полазећи од датог неодлучног фази квази-уређења,
- изучавање система неједначина и једначина са једним или два непозната неодлучна фази скупа,
- изучавање система неједначина и једначина са једном или две непознате неодлучне фази релације,
- развој алгоритама за израчунавање највећих решења наведених система неједначина и једначина.

Фокус истраживања је на успостављању теоријски конзистентног и рачунски ефикасног алгебарског оквира који омогућава даљу анализу неодлучних фази релацијских система неједначина и једначина, као и развој конкретних поступака за решавање ових система у оквиру предложеног приступа.

3. Expected results of scientific research (до 200 речи)

It is expected that the research will result in the introduction of a new algebraic framework for hesitant fuzzy sets, based on the power set quantale of a semigroup, which enables the generalization and systematization of existing approaches in fuzzy set theory and hesitant fuzzy set theory.

Most of the expected results of the proposed doctoral dissertation have already been obtained and the realization of other results is expected by the end of the dissertation. Some of the results were accepted for publication in the M21 journal. The second part of the results has been prepared and submitted for publication in the M21a journal (categorization for mathematics), and the part is in preparation.

Очекивани резултати научног истраживања

Очекује се да ће истраживање резултирати увођењем новог алгебарског оквира за неодлучне фази скупове заснованог на кванталу подскупова полугрупе, који омогућава генерализацију и систематизацију постојећих приступа у теорији фази и неодлучних фази скупова.

Већина очекиваних резултата предложене докторске дисертације већ је добијена и до завршетка дисертације очекује се и реализација других резултата. Један део резултата прихваћен је за објављивање у часопису категорије M21. Други део резултата припремљен је и поднет за публикување у часопису категорије M21a (категоризација за математику), а део је у припреми.

4. Applied scientific methods (до 300 речи)

The starting point is the method of axiomatic construction of new algebraic structures, in which hesitant fuzzy sets and the corresponding operations over them are introduced and formally defined within a quantale structure.

The method of algebraic abstraction and generalization is employed, by which classical results from fuzzy relation theory are extended to the context of hesitant fuzzy theory. In this sense, properties of quantales, in particular the residuation property, are applied in order to solve relation inequalities and equations. Within the previously defined algebraic framework, methods from relation theory are applied.

In order to solve systems of relation inequalities and equations, constructive and algorithmic methods are used, based on iterative procedures and the determination of maximal solutions within the given algebraic framework. Algorithms for computing the greatest solution of relation systems are based on the application of the Knaster–Tarski and Kleene theorems on post-fixed points of isotone functions on a complete lattice.

Logical and formal-deductive methods are also applied, by means of which the basic properties of the introduced structures are proved.

Примењене научне методе

Полазна основа је метод аксиоматске конструкције нових алгебарских структура, при чему се уводе и формално дефинишу неодлучни фази скупови и одговарајуће операције над њима над кванталном структуром.

Користи се метод алгебарске апстракције и генерализације, којим се класични резултати из теорије фази релација проширују на теорију неодлучних фази скупова. У том смислу примењују се својства квантала, посебно својство резидуације, у циљу решавања релацијских неједначина и једначина. У оквиру тако дефинисаног алгебарског окружења, примењују се методе теорије релација.

У циљу решавања система релацијских неједначина и једначина користе се конструктивне и алгоритамске методе, засноване на итеративним поступцима и одређивању максималних решења у датом алгебарском окружењу. Алгоритми за рачунање највећег решења релацијских система засновани су на примени теорема Кнастер-Тарског и Клинија о пост-фиксним тачкама изотоне функције на комплетној мрежи.

Такође се примењују методи логичке и формално-дедуктивне анализе, којима се доказују основна својства уведених структура.

5. Списак коришћене литературе (до 25 наслова)

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КРАТКО ОБРАЗЛОЖЕЊЕ ТЕМЕ (до 100 речи)

На енглеском језику:

The proposed PhD thesis introduces a new algebraic framework for modelling hesitant fuzzy sets based on the power set quantale of a semigroup, thereby extending classical fuzzy relational theory to the hesitant fuzzy setting. Within this framework, we characterize equivalence relations and quasi-orders, analyze systems of relation inequalities and equations, and develop algorithms for computing their greatest solutions.

На српском језику:

Предложена докторска дисертација уводи нови алгебарски оквир за моделовање неодлучних фази скупова који је заснован на кванталу подскупова полугрупе. Тиме се класична теорија фази релација проширује на неодлучне фази релације. У оквиру овог приступа, дајемо карактеризацију релација еквиваленције и квази-уређења, анализирамо системе релацијских неједначина и једначина, и развијамо алгоритме за израчунавање њихових највећих решења.

УЗ ЗАХТЕВ ПРИЛАЖЕМ:

- уверење о положеним испитима на докторским студијама или оверену копију дипломе о стеченом научном називу магистара наука
- копије радова, односно доказ да су радови прихваћени за објављивање

САГЛАСНОСТ ПРЕДЛОЖЕНОГ МЕНТОРА

Сагласан сам и прихватам менторство за израду предложене докторске дисертације.

Име и презиме: Мирослав Ђурић
Потпис: [Потпис]
(име и презиме и потпис предложеног ментора)

Име и презиме: Марија Мачић
Потпис: [Потпис]
(име и презиме и потпис подносиоца захтева)

Датум и место:

20.04.2026.



Универзитет у Нишу
Природно-математички факултет у Нишу

STATEMENT ON ACCEPTANCE OF MENTORING FOR DOCTORAL DISSERTATION

Hereby, I state that I am acquainted with the application for the topic of a doctoral dissertation by the candidate **Marija Tasić** under the title of:

„Hesitant fuzzy relation inequalities and equations over the power set quantale of a semigroup”

(„Неодлучне фази релацијске неједначине и једначине над кванталом подскупова полугрупе”).

Since the application for the topic of doctoral dissertation satisfies all professional, scientific, technical and further requirements stipulated by the Law on Higher Education, Statute of University in Niš and Statute of the Faculty of Sciences and Mathematics in Niš, I accept mentoring for the aforementioned doctoral dissertation.

Niš, 20.04.2026.

Miroslav Ćirić, PhD

(Signature)



Универзитет у Нишу
Природно-математички факултет у Нишу

ИЗЈАВА О ПРИХВАТАЊУ МЕНТОРСТВА У ИЗРАДИ ДОКТОРСКЕ ДИСЕРТАЦИЈЕ

Овим изјављујем да сам упознат са пријавом теме докторске дисертације кандидата **Марије Тасић** насловом:

**„Неодлучне фази релацијске неједначине и једначине над
кванталом подскупова полугрупе”**

**(„Hesitant fuzzy relation inequalities and equations over the power set
quantale of a semigroup”).**

Пошто пријава теме докторске дисертације испуњава све стручне, научне, техничке и остале захтеве дефинисане Законом о високом образовању, Статутом Универзитета у Нишу и Статутом Природно-математичког факултета у Нишу, прихватам да будем ментор у изради наведене докторске дисертације.

Ниш, 20.04.2026. године

Проф. др Мирослав Тирић

(Потпис предложеног ментора)

Data on Doctoral Dissertation	
Proposed title of the topic in Serbian	Неодлучне фази релацијске неједначине и једначине над кванталом подскупова полугрупе (превод главног наслова)
Proposed title of the topic in English	Hesitant fuzzy relation inequalities and equations over the power set quantale of a semigroup (главни наслов)
Scientific field	Natural Sciences and Mathematics
Scientific subfield	Mathematical Sciences
Narrow scientific field	Mathematical Sciences
Scientific discipline	Algebra and Mathematical Logic
Information about the candidate	
Last name, first parent name and first name	Tasić Radosav Marija
Name of the study program	PhD School of Mathematics
Information about the supervisor	
First and last name	Miroslav Ćirić
Institution where he is employed	University of Niš, Faculty of Sciences and Mathematics, Višegradska 33, P.O. Box 224, 18000 Niš

Thesis Synopsis

Subject of scientific research

Fuzzy set theory was introduced by Lotfi A. Zadeh, as a mathematical framework for modeling uncertainty, vagueness, and imprecision. Building on this foundational work, fuzzy set theory has evolved into a complex scientific discipline with numerous extensions aimed at modeling different types of uncertainty more precisely. Classical fuzzy sets represent uncertainty by assigning each element a single membership degree from the interval $[0,1]$. Although this approach has proven effective in many applications, it can be limited in situations where it is not possible to determine a unique membership value due to incomplete information, cognitive limitations, or conflicting assessments.

To address this limitation, Torra introduced hesitant fuzzy sets (HFSs) as a natural generalization of fuzzy sets that explicitly accounts for hesitation in membership assessment. Within the HFS framework, the membership degree of an element is expressed as a set of possible values from the interval $[0,1]$, called a hesitant fuzzy element, allowing the simultaneous consideration of multiple alternative membership values. This approach is particularly well suited to modeling human judgments, where hesitation arises from uncertainty, lack of confidence, or disagreement among experts.

Hesitant fuzzy sets have been successfully applied in a wide range of domains, including decision support systems, supplier selection, risk assessment, medical diagnosis, pattern recognition, and social network analysis.

A central issue in the theory of hesitant fuzzy sets is the handling of hesitant information through appropriate operations on hesitant fuzzy elements. One of the foundational approaches to defining operations on HFSs is based on lifting classical fuzzy set operations to hesitant fuzzy elements. In this framework, binary operations such as union, intersection and

complement, as well as algebraic sum and product, are defined by applying the corresponding fuzzy operation to every pair of values across the relevant hesitant fuzzy elements.

In addition, the algebraic aspect of HFS theory has also been developed. It has been shown that, under certain conditions, hesitant fuzzy sets admit a lattice structure and even residuated lattices. These algebraic structures provide a theoretical foundation for approximate reasoning and systematic decision-making under hesitation. However, they are defined only for specialized classes of hesitant fuzzy sets or for specific orderings, which indicates certain limitations in terms of applicability.

Although different types of lattice-valued fuzzy sets and relations have been studied in the literature, most existing approaches are formulated within classical residuated lattices or other fixed algebraic structures, where uncertainty is typically represented by a single membership degree. In contrast, hesitant information involves sets of possible values, which requires an algebraic framework that can naturally handle such multi-valued information while still supporting the study of relation inequalities and equations. This motivates the use of power set quantales as the underlying membership structure, since they provide a natural setting for extending fuzzy relational methods to the hesitant context, while preserving the residuation property.

The subject of this dissertation is the introduction and analysis of a new algebraic framework within HFS theory. The framework is based on the power set quantale of a semigroup and enables the extension of classical and fuzzy relational properties to the hesitant fuzzy context. One of the central aspects of the research is the characterization of hesitant fuzzy equivalence relations, their corresponding classes and partitions, as well as the characterization of quasi-orders, including the construction of an equivalence relation from a given quasi-order.

Further, the research includes the study of systems of hesitant fuzzy relation inequalities and equations, with one side containing the composition of an unknown hesitant fuzzy set and given hesitant fuzzy relations over a power set quantale, while the other side contains only the same unknown hesitant fuzzy set or another one. In particular, our research includes the characterization of solution sets of these systems and the development of algorithms for computing their greatest solutions. Moreover, the characterization of solution sets has been introduced, in terms of closure and opening systems in the quantale of hesitant fuzzy sets, together with the corresponding closure and opening operators. Finally, the subjects of our investigation have been more general non-linear systems of hesitant fuzzy relation inequalities and equations, called hesitant weakly linear systems. Two types of weakly linear systems will be analyzed within the proposed framework: homogeneous which include only fuzzy relations on a single set, and heterogeneous one composed of fuzzy relations on two possible different sets.

Aim of scientific research

Research in this area has several important objectives.

The main objective of this research is the introduction and analysis of a new algebraic framework for hesitant fuzzy sets, based on the power set quantale of a semigroup. Within the proposed framework, the objective is to develop a general theoretical setting that enables the

extension of classical and fuzzy relational methods within the HFS theory, while preserving the residuation property.

In particular, the research will focus on:

- the characterization of hesitant fuzzy equivalence relations, including the corresponding equivalence classes and partitions of the set,
- the characterization of hesitant fuzzy quasi-orders,
- the construction of a hesitant fuzzy equivalence relation starting from a given hesitant fuzzy quasi-order,
- the study of systems of inequalities and equations with one or two unknown hesitant fuzzy sets,
- the study of systems of inequalities and equations with one or two unknown hesitant fuzzy relations,
- the development of algorithms for determining the greatest solutions of the aforementioned systems of inequalities and equations.
-

The focus of the research is to establish a theoretically consistent and computationally efficient algebraic framework that enables further analysis of hesitant fuzzy systems of relation inequalities and equations, as well as to develop concrete procedures for solving these systems within the proposed framework.

Expected results of scientific research

It is expected that the research will result in the introduction of a new algebraic framework for hesitant fuzzy sets, based on the power set quantale of a semigroup, which enables the generalization and systematization of existing approaches in fuzzy set theory and hesitant fuzzy set theory.

Most of the expected results of the proposed doctoral dissertation have already been obtained and the realization of other results is expected by the end of the dissertation. Some of the results were accepted for publication in the M21 journal. The second part of the results has been prepared and submitted for publication in the M21a journal (categorization for mathematics), and the part is in preparation.

Applied scientific methods

The starting point is the method of axiomatic construction of new algebraic structures, in which hesitant fuzzy sets and the corresponding operations over them are introduced and formally defined within a quantale structure.

The method of algebraic abstraction and generalization is employed, by which classical results from fuzzy relation theory are extended to the context of hesitant fuzzy theory. In this sense, properties of quantales, in particular the residuation property, are applied in order to solve relation inequalities and equations. Within the previously defined algebraic framework, methods from relation theory are applied.

In order to solve systems of relation inequalities and equations, constructive and algorithmic methods are used, based on iterative procedures and the determination of

maximal solutions within the given algebraic framework. Algorithms for computing the greatest solution of relation systems are based on the application of the Knaster–Tarski and Kleene theorems on post-fixed points of isotone functions on a complete lattice.

Logical and formal-deductive methods are also applied, by means of which the basic properties of the introduced structures are proved.

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Подаци о докторској дисертацији	
Предлог наслова теме докторске дисертације	Неодлучне фази релацијске неједначине и једначине над кванталом подскупова полугрупе (превод главног наслова)
Предлог наслова теме на енглеском језику	Hesitant fuzzy relation inequalities and equations over the power set quantale of a semigroup (главни наслов)
Научно поље	Природно-математичке науке
Научна област	Математичке науке
Ужа научна област	Математичке науке
Научна дисциплина	Алгебра и математичка логика
Подаци о кандидату	
Презиме, име једног родитеља и име	Тасић Радосав Марија
Назив студијског програма	Докторска школа математике
Подаци о ментору	
Име и презиме	Мирослав Ћирић
Установа у којој је запослен	Универзитет у Нишу, Природно-математички факултет у Нишу, Вишеградска 33, Пош. фах: 224, 18000 Ниш

Синопис тезе

Предмет научног истраживања

Теорију фази скупова увео је Lotfi A. Zadeh, као математички оквир за моделовање неизвесности, неодређености и непреизности. Полазећи од овог фундаменталног рада, теорија фази скупова еволуирала је у комплексну научну дисциплину са бројним проширењима развијеним ради прецизнијег моделовања различитих видова неизвесности. Класични фази скупови моделују неизвесност тако што сваком елементу придружују јединствен степен припадности из интервала $[0,1]$. Иако се овај приступ показао ефикасним у бројним применама, он може бити ограничен у ситуацијама у којима није могуће одредити јединствену вредност степена припадности услед непотпуних информација, когнитивних ограничења или противречних процена.

У циљу превазилажења овог ограничења, Тогта је увео неодлучне фази скупове (НФС) као природну генерализацију фази скупова, који експлицитно моделују оклевање при процени степена припадности. У оквиру теорије неодлучних фази скупова, степен припадности елемента представљен је скупом могућих вредности из интервала $[0,1]$, који се назива неодлучни фази елемент. Тиме се омогућава истовремено разматрање више алтернативних вредности степена припадности. Овај приступ посебно је погодан за моделовање људских процена, у којима се оклевање јавља услед неизвесности, недостатка самопоуздања или несагласности међу експертима.

Неодлучни фази скупови нашли су широку примену у различитим областима, укључујући системе за подршку одлучивању, избор добављача, процену ризика, медицинску дијагностику, препознавање образаца и анализу друштвених мрежа.

Централно питање у теорији неодлучних фази скупова односи се на обраду информација са оклевањем применом одговарајућих операција над неодлучним фази елементима. Један од основних приступа дефинисању таквих операција заснива се на подизању (lifting) класичних операција дефинисаних над фази скуповима на ниво неодлучног фази елемената. У том контексту, бинарне операције као што су унија, пресек и комплемент, као и алгебарски збир и производ, дефинишу се применом одговарајућих фази операција на све парове вредности из датих неодлучних фази елемената.

Поред наведеног, развијен је и алгебарски аспект теорије неодлучних фази скупова. Показано је да неодлучни фази скупови под одређеним условима формирају структуру мреже, па чак и резидуиране мреже. Ове алгебарске структуре пружају теоријску основу за апроксимативно закључивање и систематско доношење одлука у условима оклевања. Међутим, оне су дефинисане само за специјализоване класе неодлучних фази скупова или за специфична уређења, што указује на извесна ограничења у погледу примене.

Иако су у литератури проучавани различити типови фази скупова и релација са вредностима у мрежама, већина постојећих приступа формулисана је у оквиру класичних резидуираних мрежа или других фиксних алгебарских структура, при чему је неизвесност обично представљена једним степеном припадности. Насупрот томе, информације са оклевањем подразумевају скупове могућих вредности, што природно захтева адекватан алгебарски оквир, који може да обради такве вишевредносне информације, док истовремено омогућава анализу релацијских неједначина и једначина. То нас је мотивисало да као основну структуру припадности разматрамо квантале подскупова полугрупе, будући да они представљају природан оквир за проширење фази релацијских метода на контекст неодлучних фази релација, уз очување својства резидуације.

Предмет истраживања ове дисертацији јесте увођење и анализа новог алгебарског оквира у теорији неодлучних фази скупова. Овај оквир заснива се на кванталу подскупова полугрупе и омогућава проширење класичних и фази релацијских својстава на контекст неодлучних фази релација. Један од централних аспеката истраживања јесте карактеризација неодлучних фази релација еквиваленције, њихових одговарајућих класа и партиција, као и карактеризација квази-уређења, укључујући и конструисање релације еквиваленције од датог квази-уређења.

Поред тога, истраживање обухвата проучавање система неједначина и једначина неодлучних фази релација, где једна страна садржи композицију непознатог неодлучног фази скупа и датих неодлучних фази релација над кванталом партитивног скупа, док друга страна садржи искључиво тај исти или неки други непознати неодлучни фази скуп. Специјално, наше истраживање укључује карактеризацију скупова решења ових система и развој алгоритама за израчунавање њихових највећих решења. Штавише, уведена је карактеризација скупова решења преко система затворења и отворења у кванталу неодлучних фази скупова, заједно са одговарајућим операторима затворења и отворења. Коначно, предмет нашег истраживања јесу и општији нелинеарни системи неједначина и једначина неодлучних фази релација, названи неодлучни слабо линеарни системи. У оквиру предложеног модела биће анализирана два типа слабо линеарних система:

хомогени, који обухватају само неодлучне фази релације на једном скупу, и хетерогени, састављени од неодлучних фази релација на два потенцијално различита скупа.

Циљ научног истраживања

Истраживање у овој области има неколико важних циљева.

Основни циљ овог истраживања јесте увођење и анализа новог алгебарског оквира за неодлучне фази скупове, заснованог на кванталу подскупова полугрупе. У оквиру предложеног оквира, циљ је развити општу теоријску поставку која омогућава проширење класичних и фази релацијских метода у оквиру теорије неодлучних фази скупова, уз очување својства резидуације.

Посебно, истраживање ће се усмерити на:

- карактеризацију неодлучних фази релација еквиваленције, укључујући одговарајуће класе еквиваленције и партиције скупа,
- карактеризацију неодлучних фази квази-уређења,
- конструкцију неодлучне фази релације еквиваленције полазећи од датог неодлучног фази квази-уређења,
- изучавање система неједначина и једначина са једним или два непозната неодлучна фази скупа,
- изучавање система неједначина и једначина са једном или две непознате неодлучне фази релације,
- развој алгоритама за израчунавање највећих решења наведених система неједначина и једначина.

Фокус истраживања је на успостављању теоријски конзистентног и рачунски ефикасног алгебарског оквира који омогућава даљу анализу неодлучних фази релацијских система неједначина и једначина, као и развој конкретних поступака за решавање ових система у оквиру предложеног приступа.

Очекивани резултати научног истраживања

Очекује се да ће истраживање резултирати увођењем новог алгебарског оквира за неодлучне фази скупове заснованог на кванталу подскупова полугрупе, који омогућава генерализацију и систематизацију постојећих приступа у теорији фази и неодлучних фази скупова.

Већина очекиваних резултата предложене докторске дисертације већ је добијена и до завршетка дисертације очекује се и реализација других резултата. Један део резултата прихваћен је за објављивање у часопису категорије M21. Други део резултата припремљен је и поднет за публиковање у часопису категорије M21a (категоризација за математику), а део је у припреми.

Примењене научне методе

Полазна основа је метод аксиоматске конструкције нових алгебарских структура, при чему се уводе и формално дефинишу хезитант фази скупови и одговарајуће операције над њима над кванталном структуром.

Користи се метод алгебарске апстракције и генерализације, којим се класични резултати из теорије фази релација проширују на контекст хезитант фази теорије. У том

смыслу примењују се својства квантала, посебно својство резидуације, у циљу решавања релационих неједначина и једначина. У оквиру тако дефинисаног алгебарског окружења, примењују се методе теорије релација.

У циљу решавања система релационих неједначина и једначина користе се конструктивне и алгоритамске методе, засноване на итеративним поступцима и одређивању максималних решења у датом алгебарском окружењу. Алгоритми за рачунање највећег решења релационих система засновани су на примени теорема Кнастер-Тарског и Клинија о пост-фиксним тачкама изотоне функције на комплетној мрежи.

Такође се примењују методи логичке и формално-дедуктивне анализе, којима се доказују основна својства уведених структура.

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Platilac (naziv, adresa, mesto) MARIJA TASIC GORNJA TRNICA, NEMA ULICE BB Trgovište Svrha plaćanja Uplata za prijavu teme doktorske disertacije Primalac (naziv, adresa, mesto) Prirodno-matematički fakultet Visegradska 33 Nis Datum izvršenja 17.04.2026 Hitnost: X	Šifra plaćanja Iznos 289 30.000,00 RSD Račun platioca 205-9001017051090-62 Račun primaoca 840-0000032807845-68 Model i poziv na broj odobrenja 12 <table border="1" style="width: 100%; border-collapse: collapse; margin-top: 5px;"> <tr> <td style="width: 50%;">Status</td> <td>Realizovan</td> </tr> <tr> <td>ID Transakcije</td> <td>0999F4966CD039</td> </tr> <tr> <td>Autorizovao</td> <td>MARIJA TASIC</td> </tr> <tr> <td>Vreme slanja</td> <td>17.04.2026 11:12:01</td> </tr> <tr> <td>Provizija</td> <td>15,00 RSD</td> </tr> </table> Potvrda važi bez pečata i potpisa	Status	Realizovan	ID Transakcije	0999F4966CD039	Autorizovao	MARIJA TASIC	Vreme slanja	17.04.2026 11:12:01	Provizija	15,00 RSD
Status	Realizovan										
ID Transakcije	0999F4966CD039										
Autorizovao	MARIJA TASIC										
Vreme slanja	17.04.2026 11:12:01										
Provizija	15,00 RSD										

Marija Tasić – short biography

Marija Tasić was born in Vranje, Serbia, on September 22, 1988. She completed her primary education at the „Branko Radičević” Elementary School in Trgovište, as a recipient of the „Vuk Karadžić” Diploma and as the student of the generation. In 2007, she graduated from the “Bora Stanković” Gymnasium in Vranje, also as a recipient of the „Vuk Karadžić” Diploma.

She enrolled in Bachelor’s studies in 2007 at the Faculty of Mathematics, University of Belgrade, in the module for Professor of Mathematics and Computer Science. She graduated in 2011 with an average grade of 8.05, earning the title of Bachelor of Mathematics.

She subsequently enrolled in Master’s studies in the same module. Her Master’s thesis, titled „An E-course of Functions in Programming” (in Serbian), was defended in 2012 under the supervision of Professor Miroslav Marić. She completed her Master’s studies with an average grade of 10.00, earning the title of Master of Mathematics.

In 2016, she enrolled in PhD studies at the University of Niš, Faculty of Sciences and Mathematics, within the PhD School of Mathematics, module: Algebra and Mathematical Logic.

Since 2012, she has been employed at the Pedagogical Faculty in Vranje, University of Niš, as a Teaching Associate / Teaching Assistant in the narrow scientific field of Mathematics and Informatics and Informatics and Computer Science.

Marija Tasić – List of publications

1. **M. Tasić**, M. Ćirić, J. Ignjatović, S. Stanimirović, M. Stanković, *Algebraic characterization of hesitant fuzzy relation inequalities and equations over power set quantales*, FILOMAT (2026), accepted for publication. (M21)
2. M. Stanković, **M. Tasić**, Đ. Stakić, A. Damjanović, *Exploring Wikipedia Usage in Primary School Education: Students' Experiences and Perspectives*, International Journal of Cognitive Research in Science, Engineering and Education, vol. 13, no. 2, 2025, pp. 323-334. (M22)
3. Đ. Stakić, M. Tasić, **M. Stanković**, and M. Bogdanović, *Students' Attitudes Towards the Use of Wikipedia: A Teaching Tool and a Way to Modernize Teaching*, Área Abierta, vol. 21, no. 2, 2021, pp. 309-25. (M23)
4. S. Anđelković, **M. Tasić**, A. Milanović, Educational Computer Software Aimed at Improving the Quality of Elementary Mathematics Instruction. *1st International Scientific Conference Teaching, Learning and Teacher Education (TLTE), Quality of Teaching and Learning (QTL) - Reality and Possibilities*, pp. 349-364, Pedagogical Faculty in Vranje, 2023. (M33)
5. **M. Tasić**, Креирање електронских курсева помоћу Canvas система за управљање учењем. *Имплементација нових технологија у образовни систем, зборник радова*, стр. 85-96, Педагошки факултет, Врање, 2022. (M63)
6. А. Анђелковић, М. Тасић, Дигитални рефлексивни алати у универзитетској настави и учењу. *XVIII скуп Трендови развоја: универзитетско образовање за привреду* (216-218). Копаоник: Универзитет у Новом Саду, Факултет техничких наука - Нови Сад, 2022. (M63)
7. М. Богдановић, **M. Tasić**, *Практикум из информатике*, Врање: Педагошки факултет у Врању, 2021, ISBN 978-86-6301-043-7 COBISS.SR-ID 51784969.
8. М. Богдановић, М. Станковић, **M. Tasić**, *Елементарни математички појмови – теорија и задаци*, Врање: Педагошки факултет у Врању, 2019, ISBN 978-86-6301-031-4, COBISS.SR-ID 279809804.
9. М. Богдановић, М. Станковић, **M. Јордановић**, *Some comments on isometries of sphere*. Годишњак Педагошког факултета у Врању, књига VIII, 1/2017, стр. 65-73, 2017. (M52)
10. С. Јанковић, **M. Јордановић**, *Компјутерске игре у функцији развијања почетних математичких појмова*, Шеста међународна Конференција ТИО, Зборник радова ТИО 2016, стр. 161-165, Чачак, 28-29. мај 2016; Универзитет у Крагујевцу, Факултет техничких наука, Чачак 2016. (M33)
11. М. Станковић, Ђ. Стакић, М. Богдановић, **M. Јордановић**, *Википедија као могућност за иновације у настави Информатике на примеру Учитељског факултета*, Иновације у настави XXVIII, 2015/4, стр. 114-120, 2015. (M52)
12. **M. Јордановић**, *Предности Сингапурске математике*, Могућности модификовања методике васпитно-образовног рада и методике разредне наставе, тематски зборник, стр. 161-173, Врање 2014. (M45)
13. М. Станковић, **M. Јордановић**, С. Јанковић, *Примена програмског пакета GeoGebra у циљу осавремењивања наставе математике*, Компетенције васпитача за друштво знања, тематски зборник, стр. 547-554, Кикинда 25. октобар 2014. (M53)
14. **M. Јордановић**, Н. Миленковић, *Решавање проблема трговачког путника методом симулираног каљења*, Годишњак Учитељског факултета у Врању, Учитељски факултет у Врању, стр. 51-61, Врање 2014. (M52)
15. **M. Јордановић**, М. Станковић, *Moodle платформа као софтверска подршка савременој настави*, Савремене тенденције у наставним и ваннаставним активностима на учитељским (педагошким) факултетима, тематски зборник, стр. 364-376, Врање 2014. (M45)

	УНИВЕРЗИТЕТ У НИШУ ПРИРОДНО-МАТЕМАТИЧКИ ФАКУЛТЕТ У НИШУ	Број:
		ПРИЈАВА ТЕМЕ
	УВЕРЕЊЕ О ПОЛОЖЕНИМ ИСПИТИМА	Датум:
		20.04.2026

На основу захтева који је поднела **Марија Тасић**, број индекса **2**, а на основу члана 29. Закона о општем управном поступку ("Сл. Гласник РС", бр. 18/2016, 95/2018 - аутентично тумачење и 2/2023 - Одлука УС) и чл. 162. став 2. Статута Природно-математичког факултета у Нишу, Универзитета у Нишу, Служба за наставу и студентска питања Секретаријата Факултета издаје

УВЕРЕЊЕ

да је **Марија (Радосав) Тасић**, рођена **22.09.1988.** године, место **Врање**, општина **Врање**, Република **Република Србија**, уписала **VI** годину (четврти пут), **докторске академске** студије, на студијском програму **Докторска школа математике**, школске **2025/2026** године, као **самофинансирајући** студент, положила испите из следећих предмета:

Р.бр.	Назив предмета	ЕСПБ	Оцена	
1.	Студијски истраживачки рад 1	5	10 (десет)	ПОЛОЖЕН
2.	Математичка логика	15	10 (десет)	ПОЛОЖЕН
3.	Некласичне логике	10	8 (осам)	ПОЛОЖЕН
4.	Студијски истраживачки рад 2	5	10 (десет)	ПОЛОЖЕН
5.	Општа алгебра	15	10 (десет)	ПОЛОЖЕН
6.	Теорија полугрупа	10	10 (десет)	ПОЛОЖЕН
7.	Студијски истраживачки рад 3	10	10 (десет)	ПОЛОЖЕН
8.	Теорија уређених скупова	10	6 (шест)	ПОЛОЖЕН
9.	Уређене алгебарске структуре	10	10 (десет)	ПОЛОЖЕН
10.	Студијски истраживачки рад 4	10	10 (десет)	ПОЛОЖЕН
11.	Фази скупови и системи	10	10 (десет)	ПОЛОЖЕН
12.	Теорија полупрстена	10	10 (десет)	ПОЛОЖЕН
13.	Студијски истраживачки рад 5	20	10 (десет)	ПОЛОЖЕН
Закључно са редним бројем: 13				
Укупно ЕСПБ:		140.00	Просечна оцена:	9.54 (девет, 54/100)

СЛУЖБА ЗА НАСТАВУ
И СТУДЕНТСКА ПИТАЊА





Универзитет у Нишу
Природно-математички факултет у Нишу

**STATEMENT ON REQUEST FOR APPROVAL OF A DOCTORAL DISSERTATION
TOPIC AT ANOTHER HIGHER EDUCATION INSTITUTION**

I hereby declare that the request for approval of the doctoral dissertation topic „Hesitant fuzzy relation inequalities and equations over the power set quantale of a semigroup” („Неодлучне фази релацијске неједначине и једначине над кванталом подскупова полугрупе”), which is being submitted to the Faculty of Sciences and Mathematics, University of Niš, has not been submitted, either in whole or in part, to any other higher education institution, either in the country or abroad.

Niš, 20.04.2026.

Marija Tasić

A handwritten signature in blue ink, appearing to read "Marija Tasić", is written over a horizontal line.

(Signature)



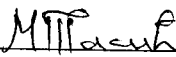
Универзитет у Нишу
Природно-математички факултет у Нишу

**ИЗЈАВА О ЗАХТЕВУ ЗА ОДОБРАВАЊЕ ТЕМЕ
ДОКТОРСКЕ ДИСЕРТАЦИЈЕ ДРУГОЈ ВИСОКОШКОЛСКОЈ УСТАНОВИ**

Изјављујем да захтев за одобравање теме докторске дисертације „Неодлучне фази релацијске неједначине и једначине над кванталом подскупова полугрупе” („Hesitant fuzzy relation inequalities and equations over the power set quantale of a semigroup”), која се пријављује на Природно-математичком факултету Универзитета у Нишу, није, ни у целини, нити у деловима, упућен другој високошколској установи, како у земљи, тако ни у иностранству.

Ниш, 20.04.2026. године

Марија Тасић


(Потпис кандидата)



Универзитет у Нишу

Природно-математички факултет у Нишу

**ИЗЈАВА О ЈЕЗИКУ НА КОМЕ ЋЕ БИТИ НАПИСАНА И ОДБРАЊЕНА
ДОКТОРСКА ДИСЕРТАЦИЈА**

Изјављујем да ће докторска дисертација, под насловом **„Неодлучне фази релацијске неједначине и једначине над кванталом подскупова полугрупе”** (енгл. **„Hesitant fuzzy relation inequalities and equations over the power set quantale of a semigroup”**), која се пријављује на Природно-математичком факултету Универзитета у Нишу, бити написана и одбрањена на **енглеском** језику.

Ниш, 20.04.2026. године

Марија Тасић

(Потпис кандидата)



Универзитет у Нишу

Природно-математички факултет у Нишу

**STATEMENT ON THE LANGUAGE IN WHICH THE DOCTORAL
DISSERTATION WILL BE WRITTEN AND DEFENDED**

I declare that the doctoral dissertation, entitled **„Hesitant fuzzy relation inequalities and equations over the power set quantale of a semigroup”** („Неодлучне фази релацијске неједначине и једначине над кванталом подскупова полугрупе”), which is applied for at the Faculty of Sciences and Mathematics, University of Niš, will be written and defended in **English**.

Niš, 20.04.2026.

Marija Tasić

A handwritten signature in blue ink, appearing to read 'MTasić', written over a horizontal line.

(Signature)



Република Србија

УУБ

Универзитет у Београду
Математички факултет, Београд



Оснивач: Република Србија

Дозволу за рад број 612-00-02666/2010-04 од 10. децембра 2010.
године је издало Министарство просвете и науке Републике Србије

Марија, Радосав, Јордановић

рођена 22. септембра 1988. године у Врању, Република Србија, уписана школске
2011/2012. године, а дана 23. октобра 2012. године завршила је мастер
академске студије, другој степена, на студијском програму Математика,
обима 60 (шездесет) бодова ЕСПБ са просечном оценом 10,00 (десет и 0/100).

На основу тога издаје јој се ова диплома о сачењеном високом образовању и академском називу
мастер математичар

Број: 1240600

У Београду, 28. марта 2013. године

Декан

Проф. др Милош Матељевић

Ректор

Проф. др Владимир Бумбаширевић

00012496

Република Србија
ЈАВНИ БЕЛЕЖНИК
Оливера Стаменовић
Врање
Цара Душана 3

УОП - I:1413-2026
страна 1 (један)

Потврђује се да је ова копија истоветна са копираном исправом која је написана на компјутерском штампачу и која се састоји од 1 (једној) стране/а.-----
Накнада за оверу 1 (један) примерка наплаћена је у укупном износу од 648,00 (шестсто четрдесет осам динара) са урачунатим ПДВ-ом на основу члана 21 тарифног броја 10 Јавнобележничке тарифе.-----

ЈАВНИ БЕЛЕЖНИК
Оливера Стаменовић
Врање
Цара Душана 3

За јавног бележника
Јавнобележнички
сарадник
Јасмина Пашалић
број решења: 4103-2-IV-
8/2021
од 18.11.2021 год.

УОП - I:1413-2026

Дана 17.04.2026. (седамнаестог априла две хиљаде двадесетшесте) године, у 08:39 (осам часова и тридесет девет минута), у Врању, оверено у 1 (један) примерак/ка за потребе странке.

(потпис)





Република Србија

УУБ

Универзитет у Београду
Математички факултет, Београд



Оснивач: Република Србија

Дозволу за рад број 612-00-02666/2010-04 од 10. децембра 2010.
године је издало Министарство просвете и науке Републике Србије

Диплома

Марија, Радосав, Јордановић

рођена 22. септембра 1988. године у Врању, Република Србија, уписана школске
2007/2008. године, а дана 12. јула 2011. године завршила је основне академске студије,
првог степена, на студијском програму Професор математике и рачунарства,
одима 240 (двеста четрдесет) бодова ЕСПБ са просечном оценом 8,05 (осам и 5/100).

На основу тога издаје јој се ова диплома о сачењеном високом образовању и стручном називу
дипломирани математичар

Број: 1249500

У Београду, 28. марта 2013. године

Декан
Проф. др Миодраг Матељевић

Миодраг Матељевић

Ректор
Проф. др Владимир Бумбаширевић

Владимир Бумбаширевић

00012488

Република Србија
ЈАВНИ БЕЛЕЖНИК
Оливера Стаменовић
Врање
Цара Душана 3

УОП - I:1414-2026
страна 1 (један)

Потврђује се да је ова копија истоветна са копираном исправом која је написана на компјутерском штампачу и која се састоји од 1 (једној) стране/а.-----
Накнада за оверу 1 (један) примерка наплаћена је у укупном износу од 648,00 (шестсто четрдесет осам динара) са урачунатим ПДВ-ом на основу члана 21 тарифног броја 10 Јавнобележничке тарифе.-----

ЈАВНИ БЕЛЕЖНИК
Оливера Стаменовић
Врање
Цара Душана 3

За јавног бележника
Јавнобележнички
сарадник
Јасмина Пашалић
број решења: 4103-2-IV-
8/2021
од 18.11.2021 год.

УОП - I:1414-2026

Дана 17.04.2026. (седамнаестог априла две хиљаде двадесетшесте) године, у 08:39 (осам часова и тридесет девет минута), у Врању, оверено у 1 (један) примерак/ка за потребе странке.

(потпис)



МКР



РЕПУБЛИКА СРБИЈА

БРАЊЕ



ГРАД _____

ОПШТИНА _____

ИЗВОД ИЗ МАТИЧНЕ КЊИГЕ РОЂЕНИХ

У матичну књигу рођених која се води за матично подручје **БРАЊЕ**, под текућим бројем _____
за годину **1988** _____ је упис рођења:

Име	МАРИЈА	
Презиме	ЈОРДАНОВИЋ	(пол)
Дан, месец, година и час рођења	22. (двадесетдруги) септембар 1988. године у 18,00	
Место и општина-град рођења, а ако је лице рођено у иностранству и држава	БРАЊЕ, БРАЊЕ	
Јединствени матични број грађана	2209988747037	
Држављанство	РЕПУБЛИКЕ СРБИЈЕ	
Подаци о родитељима	оца	мајке
Име	РАДОСАВ	СУНЧИЦА
Презиме (и презиме пре закључења брака)	ЈОРДАНОВИЋ	ЈОРДАНОВИЋ
Јединствени матични број грађана		
Дан, месец и година рођења		
Место и општина-град рођења, а ако је лице рођено у иностранству и држава	06.01.1958. Г. ТРГОВИШТЕ	18.04.1959. Г. ГЛУЖА, ТРГОВИШТЕ
Држављанство	РЕПУБЛИКЕ СРБИЈЕ	РЕПУБЛИКЕ СРБИЈЕ
Пребивалиште и адреса	ТРГОВИШТЕ,	ТРГОВИШТЕ,
Накнадни уписи и забелешке:		
Број: _____ у БРАЊУ дана 19.10.2016. 1 5247810 (М.П.) Потпис матичара БРАНИСЛАВ КОСТИЋ		

Република Србија
ЈАВНИ БЕЛЕЖНИК
Оливера Стаменовић
Врање
Цара Душана 3

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Your submission #808 has been accepted to FILOMAT

2026-03-25 17:23

Dear Marija Tasić,

I am pleased to inform you that we have decided to accept your submission without further revision. After careful review, we found your submission, *Algebraic characterization of hesitant fuzzy relation inequalities and equations over power set quantales* (ID #808), to meet or exceed our expectations. We are excited to publish your piece in FILOMAT, and we thank you for choosing our journal as a venue for your work.

Your submission is now forthcoming in a future issue of FILOMAT, and you are welcome to include it in your list of publications. We recognize the hard work that goes into every successful submission, and we want to congratulate you on reaching this stage.

Your submission will now undergo copy editing and formatting to prepare it for publication. Please upload your LaTeX file of the final version of your paper.

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Kind regards,

Dragan S. Djordjević

University of Niš
Faculty of Sciences and Mathematics



Study Research Work 5

**Algebraic characterization of hesitant
fuzzy relation inequalities and equations
over power set quantales**

Supervisor:
Prof. Miroslav Ćirić

Student:
Marija Tasić

Niš, 2026.

Contents

1	Introduction	2
2	Preliminaries	4
2.1	Power set quantales and related structures	4
2.2	Hesitant fuzzy sets over the power set quantale	9
2.3	Additional properties and operations on hesitant fuzzy sets	11
3	Hesitant fuzzy relations over the power set quantale	12
3.1	Hesitant fuzzy relations with specific properties	15
4	Hesitant fuzzy relation inequalities	19
5	Hesitant fuzzy relation equations	25
6	Algebraic characterizations of the sets of solutions	31
7	Conclusion	35

1 Introduction

Since the seminal work of Zadeh introducing fuzzy sets as a mathematical framework for modeling uncertainty, vagueness and imprecision, fuzzy set theory has evolved into a rich and diverse field with numerous extensions designed to capture different facets of uncertainty [34]. Classical fuzzy sets describe uncertainty by assigning to each element a single membership degree in the unit interval $[0, 1]$. While this representation has proven effective in many applications, it may be overly restrictive in situations where decision-makers are unable to specify a unique membership value due to incomplete information, cognitive limitations, or conflicting evaluations.

To address this limitation, hesitant fuzzy sets (HFSs) were introduced by Torra [25] as a natural generalization of fuzzy sets that explicitly accounts for hesitation in membership assessment. In an HFS, the membership degree of an element is represented by a set of possible values in $[0, 1]$, called a *hesitant fuzzy element*, reflecting the fact that several membership degrees may be considered plausible. This approach is particularly suitable for modeling human judgments, where hesitation often arises from uncertainty, lack of confidence, or disagreement among experts. The introduction of hesitant fuzzy sets has led to extensive research on their mathematical properties and associated tools. Various operations, aggregation operators, distance and similarity measures, and ranking methods for hesitant fuzzy elements have been developed [29, 31]. These theoretical advances have enabled the incorporation of hesitant fuzzy information into established methodologies, particularly in multi-criteria decision-making and group decision-making frameworks, where hesitation and disagreement among experts are common [30]. In terms of applications, hesitant fuzzy sets have been successfully applied in a wide range of domains, including decision support systems, supplier selection, risk assessment, medical diagnosis, pattern recognition, and social network analysis [28, 32, 33].

A central issue in the theory of hesitant fuzzy sets is the handling of hesitant information through appropriate operations on hesitant fuzzy elements. One of the foundational approaches to defining operations on HFSs is based on lifting classical fuzzy set operations to hesitant fuzzy elements [25, 26]. In this framework, binary operations such as union, intersection and complement, as well as algebraic sum and product, are defined by applying the corresponding fuzzy operation to every pair of values across the relevant hesitant fuzzy elements. This lifted operation methodology ensures that the resulting hesitant fuzzy element represents all possible outcomes of combining the original hesitations, which offers high-resolution representation of uncertainty compared to approaches that reduce hesitant information immediately via aggregation. Beyond pointwise operations, researchers have investigated the algebraic structure of hesitant fuzzy sets, and in [14, 19], it was shown how hesitant fuzzy sets can be endowed with a lattice structure, and even with the structure of a residuated lattice. These algebraic structures provide a theoretical basis for approximate reasoning and systematic decision-making under hesitation. While lifted operations preserve all possible membership interactions, decision-making contexts often require condensed representations of hesitant information. Aggregation operators designed for HFSs compute representative aggregated values from a hesitant set of membership degrees. These operators generalize classical aggregation methods to the hesitant context and support the synthesis of multi-attribute or multi-expert information. Another line of development involves defining ranking functions, distance measures, and similarity indices for hesitant fuzzy elements. These tools facilitate ordering and comparison between HFEs, which is essential when decision alternatives must be evaluated or clustered.

Although different types of lattice-valued fuzzy sets and relations have been studied in the literature, most existing approaches are formulated within classical residuated lattices or other fixed algebraic structures, where uncertainty is typically represented by a single membership degree. The main flow of fuzzy relation equations is that they frequently do not have solutions (meaning no solution exists for the given data). In contrast, hesitant fuzzy information involves

sets of possible values, which requires an algebraic framework that can naturally handle such multi-valued information while still supporting the study of relation inequalities and equations. For that reason, hesitant fuzzy relation equations provide a much more flexible solution space because the system can calculate a range of solutions when shifting from a single value to sets of values. This motivates the use of power set quantales as the underlying membership structure, since they provide a natural setting for extending fuzzy relational methods to the hesitant context, while preserving the residuation property needed for solving relation inequalities and equations.

Both approaches in [24] and in the present paper are based on the same framework and differ from earlier approaches in the literature. Namely, the hesitant fuzzy sets we consider here take their membership values in a monoid $S = \langle S, *, 1_* \rangle$. In other words, a hesitant fuzzy set on X is defined as a mapping $h : X \rightarrow \mathcal{P}(S)$ that assigns to each element $x \in X$ a set $h(x) \subseteq S$ of possible membership values of x . In the general case, we take S to be an arbitrary monoid, although we always have in mind concrete interpretations in which this monoid is the multiplicative reduct of a complete residuated lattice, where the multiplication $*$ is used to model strong conjunction, or where $S = [0, 1]$ and the multiplication $*$ is a t-norm on S . By applying the operation lifting method to the multiplication $*$, one obtains an operation $\otimes$ on subsets of S such that $\langle \mathcal{P}(S), \otimes, 1_\otimes \rangle$ forms a monoid with the identity $1_\otimes = \{1_*\}$. In addition, this monoid is *residuated*, in the sense that there exist operations $\backslash$ and $/$ on $\mathcal{P}(S)$ satisfying the *residuation property* given below in formula (20), and if the multiplication $*$ is commutative, then there exists an operation $\rightarrow$ on $\mathcal{P}(S)$ satisfying the *residuation property* given in (21). This well-known fact constitutes the starting point of our investigations carried out in [24] and in the present paper, where we study the significant role of residuation in $\mathcal{P}(S)$ in the development of the theory of hesitant fuzzy relations and in solving inequalities and equations involving hesitant fuzzy sets and relations. Since residuation in $\mathcal{P}(S)$ is based on set-theoretical inclusion as the ordering relation on $\mathcal{P}(S)$, it is natural to base the operations on hesitant fuzzy sets on set-theoretical union and intersection, and thus $\mathcal{P}(S)$ is here regarded as a unital quantale $\langle \mathcal{P}(S), \cup, \cap, \otimes, \emptyset, S, 1_\otimes \rangle$. In our forthcoming research we will also consider other operations on hesitant fuzzy sets that are more closely related to standard fuzzy operations on the real unit interval or operations in complete residuated lattices. However, in the case where $S = [0, 1]$ and the multiplication $*$ is taken to be a continuous t-norm on $[0, 1]$, we show that even such a set-theoretical residuation on $\mathcal{P}(S)$ is closely related to the residual implication corresponding to this continuous t-norm (see Remark 2.2).

The main objective of this paper is to provide characterizations of solution sets and to develop algorithms for solving systems consisting of inequalities of the form $w \circ R \subseteq w$ and $R \circ w \subseteq w$, as well as systems consisting of equations of the form $w \circ R = w$ and $R \circ w = w$, where R is a given hesitant fuzzy relation, $\circ$ denotes the composition of hesitant fuzzy sets and relations, and w is an unknown hesitant fuzzy set. Such equations and inequations arise from the study of fixed-point-type problems in fuzzy set theory, where a fuzzy set is related to its image under a fuzzy relational operator. They were first studied in papers of E. Sanchez [22, 23], where they were called *eigen fuzzy set equations and inequalities*, and their solutions were called *eigen fuzzy sets*. Although many other authors later used Sanchez's term eigen fuzzy set, the literature also features terms such as fixed points of fuzzy relational operators, invariant fuzzy sets, eigen solutions to fuzzy relation equations, and others. Eigen fuzzy set equations generalize classical eigenvalue problems from linear algebra to the fuzzy and nonlinear setting and play a key role in understanding the long-term behavior and structural properties of fuzzy systems. Eigen fuzzy set inequations are often more flexible and easier to solve than equations, and they provide important bounds for the solution sets of eigen equations. From an order-theoretic perspective, they correspond to post-fixed points of fuzzy relational operators. Eigen fuzzy set equations and inequalities are widely used in medical diagnosis, for identifying stable symptom-disease

relationships, in image reconstruction, for finding invariant features, in decision making, for evaluating the effectiveness of actions, and in other areas.

The solution set of the system of inequalities $w \circ R_i \subseteq w$ ($i \in I$) is characterized in Theorem 4.1. The main result is that the solution set of this system is equal to the solution set of a single inequality $w \circ F \subseteq w$, as well as to the solution set of a single equation $w \circ F = w$, where F is the least fuzzy quasi-order containing all the hesitant fuzzy relations R_i ($i \in I$). It is also shown that the solution set of this system consists of all linear combinations of F -aftersets. Similar results were obtained for the system $R_i \circ w \subseteq w$ ($i \in I$) (Theorem 4.2), as well as for the system consisting of inequalities of both types (Theorem 4.3). In Theorem 4.7, it is proven that the solution sets of all these systems form both closure and opening systems in the quantale of hesitant fuzzy sets, while in Proposition 4.8 the corresponding closure and opening operators are characterized. As a direct consequence of Proposition 4.8, an algorithm is provided for computing the greatest solution to a system $w \circ R_i \subseteq w$ ($i \in I$) contained in a given hesitant fuzzy set g . Theorem 5.7 provides an algorithm for computing the greatest solution to the system of equations $w \circ R_i = w$ ($i \in I$) contained in a given hesitant fuzzy set g , whereas Theorem 5.8 provides a different algorithm for computing the greatest solution to a single equation $w \circ R = w$ contained in g . The results presented in Section 6 provide algebraic characterizations of the solution sets of the equations $w \circ F = w$ and $F \circ w = w$, where F is an arbitrary hesitant fuzzy quasi-order.

The paper is organized as follows. In Section 2, we recall basic notions and preliminary results needed throughout the paper. Section 3 is devoted to introducing hesitant fuzzy relations over the power set quantale and investigating their basic properties. In Section 4, hesitant fuzzy relation inequalities, where the unknown is a hesitant fuzzy set, are defined, and the corresponding sets of solutions are analyzed. We show that these solution sets form both opening and closure systems, and we present an algorithm for computing the greatest solution to a system of inequalities $w \circ R_i \subseteq w$ ($i \in I$), contained in a given hesitant fuzzy set. In Section 5, we analyze hesitant fuzzy relation equations and provide a comprehensive description of their sets of solutions. We present two algorithms: one for computing the greatest solution to the system of hesitant fuzzy relation equations $w \circ R_i = w$ ($i \in I$) and another for computing the greatest solution to the single hesitant fuzzy equation $w \circ R = w$, each contained in a given hesitant fuzzy set. In Section 6, we analyze families of hesitant fuzzy sets that form the solution sets of hesitant fuzzy relation equations, where the relation is a hesitant fuzzy quasi-order.

2 Preliminaries

In this section, we present some basic notions that we use throughout the paper. The terminology and fundamental concepts are based on references [2, 3, 4, 5, 9, 11]. For further details on lattices and related structures, we refer the reader to [1, 20]. For further information about the application of quantales, we refer to [21, 27].

2.1 Power set quantales and related structures

Throughout this work, membership values are taken from a power set quantale, which provides a suitable algebraic framework for our setting. Recall that a *lattice* may be described either as a partially ordered set $\langle S, \leq \rangle$ in which every finite subset has both the least upper bound (supremum) and the greatest lower bound (infimum), or equivalently, as an algebraic structure $\langle S, \vee, \wedge \rangle$ equipped with join and meet operations that are commutative, associative, and satisfy the absorption laws. Moreover, a lattice is said to be *complete* if every subset of S , including infinite ones, has both a supremum and an infimum. A lattice is called *bounded* if it contains the greatest element (denoted by 1) and the least element (denoted by 0).

As a preliminary notion, recall that a *semigroup* is an ordered pair $\langle S, * \rangle$, where S is a non-empty set and $*$ is an associative binary operation on S . A *monoid* is an algebraic structure $\langle S, *, 1_* \rangle$ in which $\langle S, * \rangle$ forms a semigroup and 1_* is an identity element, satisfying $1_* * a = a * 1_* = a$ for all $a \in S$.

A (partially) *ordered semigroup* is a structure $\langle S, *, \leq \rangle$, where $\langle S, * \rangle$ is a semigroup and $\langle S, \leq \rangle$ is a (partially) ordered set, such that the order is compatible with multiplication, i.e., for all $a, b, c \in S$:

$$a \leq b \quad \Rightarrow \quad c * a \leq c * b \quad \text{and} \quad a * c \leq b * c.$$

If S additionally contains an identity element 1_* , then the structure $\langle S, *, 1_*, \leq \rangle$ is called an (ordered) *monoid*.

Definition 2.1. A *quantale* is an algebra $\mathcal{Q} = \langle Q, \vee, \wedge, \otimes, 0, 1 \rangle$ satisfying the following conditions:

- (Q1) $\langle Q, \vee, \wedge, 0, 1 \rangle$ is a complete lattice in which 0 is the least element and 1 is the greatest element;
- (Q2) $\langle Q, \otimes \rangle$ is a semigroup, which does not have to be commutative;
- (Q3) for arbitrary $a \in Q$ and $\{b_i\}_{i \in I} \subseteq Q$ the following infinite distributive laws hold:

$$a \otimes \left(\bigvee_{i \in I} b_i \right) = \bigvee_{i \in I} (a \otimes b_i), \quad \left(\bigvee_{i \in I} b_i \right) \otimes a = \bigvee_{i \in I} (b_i \otimes a). \quad (1)$$

The binary operation $\otimes$ is called the multiplication. If $\otimes$ is commutative on Q , then Q is called a *commutative quantale*. If the semigroup $\langle Q, \otimes \rangle$ has an identity element, that is, if there exists an element $1_\otimes \in Q$ such that $\langle Q, \otimes, 1_\otimes \rangle$ is a monoid, then Q is called a *unital quantale* and $1_\otimes$ is called the *multiplicative identity* of Q . If, moreover, $1_\otimes$ coincides with the greatest element 1, then Q is called an *integral quantale*. The least element 0 is the zero of the semigroup $\langle Q, \otimes \rangle$, i.e., $a \otimes 0 = 0 \otimes a = 0$ for every $a \in Q$. A commutative integral quantale is called a *complete residuated lattice*.

A quantale Q admits two binary operations $\backslash$ and $/$ on its carrier Q , defined as:

$$a \backslash b = \bigvee \{x \in Q \mid a \otimes x \leq b\}, \quad (2)$$

$$b / a = \bigvee \{x \in Q \mid x \otimes a \leq b\}, \quad (3)$$

for any $a, b \in Q$. We call $a \backslash b$ the *right residual of b by a*, and b / a the *left residual of b by a*. The left residual b / a is isotone in its first argument and antitone in its second argument, while the right residual $a \backslash b$ is antitone in its first argument and isotone in its second argument. The following *residuation property* holds:

$$a \otimes b \leq c \quad \Leftrightarrow \quad a \leq c / b \quad \Leftrightarrow \quad b \leq a \backslash c, \quad (4)$$

for each $a, b, c \in Q$. Moreover, for all $a, b, c \in Q$, the following implications are valid:

$$b \leq c \quad \Rightarrow \quad a \otimes b \leq a \otimes c, \quad b \otimes a \leq c \otimes a, \quad a \backslash b \leq a \backslash c, \quad b / a \leq c / a.$$

If $\otimes$ is a commutative operation, then the left and right residuals coincide and are denoted by $\rightarrow$. The operation $\rightarrow$, called the *residuum*, is antitone in its first argument and isotone in its second argument. Under this convention, the residuation property takes the following form:

$$a \otimes b \leq c \quad \Leftrightarrow \quad a \leq b \rightarrow c, \quad (5)$$

for all $a, b, c \in Q$. Furthermore, in this commutative setting, we define a new binary operation on Q , called the *biresiduum*, by:

$$a \leftrightarrow b \equiv (a \rightarrow b) \wedge (b \rightarrow a),$$

and a unary operation, called *negation*, by:

$$\neg a \equiv a \rightarrow 0,$$

for all $a, b \in Q$.

In the case when Q is a unital quantale with the multiplicative identity $1_\otimes$, then for an arbitrary family $\{b_i\}_{i \in I} \subseteq Q$ and any elements $a, b, c \in Q$, the following properties hold (cf. [27]):

$$0 \rightarrow a = 1, \tag{6}$$

$$a \rightarrow 1 = 1, \tag{7}$$

$$1_\otimes \rightarrow a = a, \tag{8}$$

$$c \rightarrow (b \rightarrow a) = (b \otimes c) \rightarrow a, \tag{9}$$

$$a \otimes (a \rightarrow b) \leq b, \tag{10}$$

$$(a \rightarrow b) \otimes c \leq a \rightarrow (b \otimes c), \tag{11}$$

$$(a \rightarrow b) \otimes (b \rightarrow c) \leq (a \rightarrow c), \tag{12}$$

$$a \rightarrow \left(\bigwedge_{i \in I} b_i \right) = \bigwedge_{i \in I} (a \rightarrow b_i), \tag{13}$$

$$\left(\bigvee_{i \in I} b_i \right) \rightarrow a = \bigwedge_{i \in I} (b_i \rightarrow a), \tag{14}$$

$$\bigvee_{i \in I} (a \rightarrow b_i) \leq a \rightarrow \left(\bigvee_{i \in I} b_i \right), \tag{15}$$

$$a \leq b \Rightarrow \neg b \leq \neg a. \tag{16}$$

In the sequel, we provide an example of a quantale that is presented in the papers [12, 16]. Kruml and Paseka in [16] referred to this example as the most general example that leads to any quantale. Example 2.1 is particularly significant, as it serves as a starting point for the main results of the paper.

Example 2.1. Let $\langle S, * \rangle$ be a semigroup, and let $\mathcal{P}(S)$ denotes the powerset of S , ordered by the set inclusion. Then, $\langle \mathcal{P}(S), \subseteq \rangle$ is a complete lattice. Define a multiplication operation $\otimes$ on $\mathcal{P}(S)$ by

$$A \otimes B = \{a * b \mid a \in A \text{ and } b \in B\}, \tag{17}$$

for all $A, B \in \mathcal{P}(S)$. Then $\mathcal{P}(S) = \langle \mathcal{P}(S), \cup, \cap, \otimes, \emptyset, S \rangle$ forms a quantale, known as the *power set quantale of S* . It is clear that $\mathcal{P}(S)$ is a commutative quantale if and only if $\langle S, * \rangle$ is a commutative semigroup, and $\mathcal{P}(S)$ is a unital quantale if and only if $\langle S, * \rangle$ is a monoid. In the latter case, the multiplicative identity in $\mathcal{P}(S)$ is $1_\otimes = \{1_*\}$, where 1_* is the identity element of the monoid $\langle S, * \rangle$. For arbitrary $B, C \in \mathcal{P}(S)$, the right residual $A \backslash B$ of B by A and the left residual B / A of B by A are respectively defined as:

$$A \backslash B = \{x \in S \mid A \otimes \{x\} \subseteq B\}, \tag{18}$$

$$B / A = \{x \in S \mid \{x\} \otimes A \subseteq B\}. \tag{19}$$

The following *residuation property* holds for every $A, B, C \in \mathcal{P}(S)$:

$$A \otimes B \subseteq C \Leftrightarrow A \subseteq C/B \Leftrightarrow B \subseteq A \backslash C. \quad (20)$$

If $\langle S, * \rangle$ is a commutative semigroup, then $\mathcal{P}(S)$ is a commutative power set quantale and the right and left residuals coincide. In that case, there is only one *residuum operation* denoted by $\rightarrow$, that is, $A \rightarrow B = A \backslash B = B/A$, for all $A, B \in \mathcal{P}(S)$, and the residuation property then becomes

$$A \otimes B \subseteq C \Leftrightarrow A \subseteq B \rightarrow C, \quad (21)$$

for all $A, B, C \in \mathcal{P}(S)$. The *biresiduum operation* $\leftrightarrow$ is then defined by

$$A \leftrightarrow B = (A \rightarrow B) \cap (B \rightarrow A), \quad (22)$$

for all $A, B \in \mathcal{P}(S)$.

Remark 2.1. It is easy to verify that $A \backslash B$ is the greatest solution to the inequality $A \otimes X \subseteq B$ in $\mathcal{P}(S)$, and B/A is the greatest solution to the inequality $X \otimes A \subseteq B$, where X is an unknown taking values in $\mathcal{P}(S)$. In other words, an alternative definition of the residuals in $\mathcal{P}(S)$ is

$$\begin{aligned} A \backslash B &= \bigcup \{X \in \mathcal{P}(S) \mid A \otimes X \subseteq B\}, \\ B/A &= \bigcup \{X \in \mathcal{P}(S) \mid X \otimes A \subseteq B\}, \end{aligned}$$

for all $A, B \in \mathcal{P}(S)$. However, we stick to the original definitions (18) and (19) of the residuals in power set quantales for the sake of simplicity.

In what follows, we introduce several specific examples of power set quantales. We start from the unit interval $S = [0, 1]$ equipped with a t-norm $*$. Then, the structure $\langle S, *, 1 \rangle$ forms a commutative monoid. By Example 2.1, it follows that the structure $\langle \mathcal{P}(S), \cup, \cap, \otimes, \emptyset, S \rangle$ is a commutative unital power set quantale, where $\otimes$ is defined by (17), and the unit element is $1_{\otimes} = \{1\}$. The associated residual operations, given in (18) and (19), coincide due to the commutativity of $\otimes$, so that $B \backslash A = A/B$ holds for all $A, B \in \mathcal{P}(S)$.

The above construction can be specialized by fixing particular t-norms on S . In this way, different subclasses of power set quantales arise:

- The *Gödel power set quantale* is obtained by taking $a * b = \min\{a, b\}$, for every $a, b \in S$;
- The *Lukasiewicz power set quantale* corresponds to $a * b = \max\{0, a + b - 1\}$, for every $a, b \in S$;
- The *product power set quantale* is induced by $a * b = a \cdot b$, for every $a, b \in S$.

In the following remark, we describe certain connections between the residuum operation on the semigroup S and the residuum operation on the power set quantale of S , in the case where S is a semigroup with carrier set $[0, 1]$ and multiplication given by a left-continuous t-norm on $[0, 1]$.

Remark 2.2. Let $\mathcal{P}(S)$ be the power set quantale of a semigroup S . Then, for arbitrary $a, b \in S$, we have that

$$\begin{aligned} \{a\} \backslash \{b\} &= \{x \in S \mid \{a\} \otimes \{x\} \subseteq \{b\}\} = \{x \in S \mid a * x = b\}, \\ \{b\} / \{a\} &= \{x \in S \mid \{x\} \otimes \{a\} \subseteq \{b\}\} = \{x \in S \mid x * a = b\}. \end{aligned}$$

In other words, we have that $\{a\} \backslash \{b\}$ is the set of all solutions to the equation $a * x = b$ in S , whereas $\{b\} / \{a\}$ is the set of all solutions to the equation $x * a = b$ in S .

In what follows, let $S = [0, 1]$, let $*$ be an arbitrary left-continuous t-norm on $[0, 1]$, and let $\rightarrow$ be the residuum operation (residual implication) on $[0, 1]$ corresponding to $\otimes$. Since the left-continuous t-norm $\otimes$ is a commutative operation on $S = [0, 1]$, it follows that the multiplication operation in $\mathcal{P}([0, 1])$ is also commutative, and hence the left and right residuum operations on $\mathcal{P}([0, 1])$ coincide. However, in this case we will not denote this operation by $\rightarrow$, as noted above, but we will retain the notation $\setminus$ (or $/$), in order to clearly distinguish it from the residuum operation on $[0, 1]$, for which we already use the symbol $\rightarrow$.

As known, for arbitrary $a, b \in S$ we have that $a \rightarrow b$ is the greatest solution of the inequality $a \otimes x \leq b$, and according to [8, Lemma 2], if the set $\{a\} \setminus \{b\} = \{b\} / \{a\}$ is non-empty, i.e., if the equation $a \otimes x = b$ is solvable in $[0, 1]$, then $a \rightarrow b$ is also the greatest solution to this equation, that is, $a \rightarrow b$ is the greatest element of the set $\{a\} \setminus \{b\}$.

If the set $\{a\} \setminus \{b\}$ is empty, that is, if the equation $a \otimes x = b$ is not solvable, then the relationship between $\{a\} \setminus \{b\}$ and $a \rightarrow b$ is not so clear. However, if $\otimes$ is a continuous t-norm, this relationship can be made considerably clearer. Recall that a continuous t-norm $\otimes$ satisfies the *divisibility condition*

$$a \otimes (a \rightarrow b) = a \wedge b, \quad (23)$$

for all $a, b \in [0, 1]$, whose equivalent form is

$$b \leq a \Rightarrow (\exists x \in [0, 1]) a \otimes x = b, \quad (24)$$

for all $a, b \in [0, 1]$ (cf. [4, Theorem 2.36 and Lemma 2.90] or [5, Lemma 1.39]). Now, according to (24) and the fact that the interval $[0, 1]$ is linearly ordered, for arbitrary $a, b \in [0, 1]$ we have that

$$b \leq a \Leftrightarrow \{a\} \setminus \{b\} \neq \emptyset \quad \text{and} \quad a < b \Leftrightarrow \{a\} \setminus \{b\} = \emptyset,$$

and therefore

$$a \rightarrow b = \begin{cases} \top(\{a\} \setminus \{b\}) & \text{if } \{a\} \setminus \{b\} \neq \emptyset, \\ 1 & \text{if } \{a\} \setminus \{b\} = \emptyset, \end{cases}$$

where $\top X$ denotes the greatest element of a set X .

It is interesting to observe that for $a \neq b$, the equation $a \otimes x = b$ has no solution in $[0, 1]$ if and only if all elements of $[0, 1]$ are solutions to the inequality $a \otimes x \leq b$.

An algebra $\mathcal{S} = \langle S, \oplus, \otimes, 0, 1 \rangle$ is called a *semiring* if $\langle S, \oplus, 0 \rangle$ is a commutative monoid, $\langle S, \otimes, 1 \rangle$ is a monoid, multiplication distributes over addition on both sides, and 0 is absorbing with respect to multiplication. A *left $\mathcal{S}$ -semimodule* is a commutative monoid $\langle M, +, 0 \rangle$ equipped with an external multiplication $S \times M \rightarrow M$, denoted by $(\lambda, x) \mapsto \lambda x$, called *left scalar multiplication*, satisfying the following axioms for all $\lambda, \mu \in S$ and all $a, b \in M$:

$$(\lambda \otimes \mu)a = \lambda(\mu a), \quad (25)$$

$$\lambda(a + b) = \lambda a + \lambda b, \quad (26)$$

$$(\lambda \oplus \mu)a = \lambda a + \mu a, \quad (27)$$

$$1a = a, \quad (28)$$

$$\lambda 0 = 0a = 0. \quad (29)$$

A *right $\mathcal{S}$ -semimodule* is defined analogously, where the external multiplication is a map $M \times S \rightarrow M$, written $(x, \lambda) \mapsto x\lambda$ and called *right scalar multiplication*, and the dual versions of conditions (25)–(29) are required to hold. An *$\mathcal{S}$ -bisemimodule* is a commutative monoid $\langle M, +, 0 \rangle$ that is simultaneously a left and a right $\mathcal{S}$ -semimodule, and for all $\lambda, \mu \in S$ and $a \in M$ the following is true:

$$(\lambda a)\mu = \lambda(\mu a). \quad (30)$$

Furthermore, if the left and right scalar multiplications coincide, i.e., if $\lambda x = x\lambda$, for all $\lambda \in S$ and $a \in M$, then $\langle M, +, 0 \rangle$ is a *strong S -bisemimodule*.

If $\mathcal{Q} = \langle Q, \vee, \wedge, \otimes, 0, 1 \rangle$ is a unital quantale with the multiplicative identity $1_\otimes$, then $\mathcal{Q}^* = \langle Q, \vee, \otimes, 0, 1_\otimes \rangle$ is a semiring, called the *semiring reduct* of $\mathcal{Q}$. For further information on semirings, we refer the reader to [10, 11]. A detailed study of algebras arising from Łukasiewicz's logic and their corresponding semiring reducts can be found in the article of Di Nola and Gerla [18].

Let $\langle P, \leq \rangle$ be a partially ordered set and let $\varphi : P \rightarrow P$ be a function. The function φ is said to be *isotone* if $a \leq b$ implies $\varphi(a) \leq \varphi(b)$ for all $a, b \in P$. If $a \leq \varphi(a)$ holds for every $a \in P$, φ is called *extensive*; if $\varphi(a) \leq a$, it is called *intensive*. A function φ is *idempotent* if it satisfies $\varphi(\varphi(a)) = \varphi(a)$ for all $a \in P$. A function φ that is isotone, extensive, and idempotent is known as a *closure operator* on $\langle P, \leq \rangle$, and any element $a \in P$ such that $\varphi(a) = a$ is called φ -closed. Dually, an *opening operator* is a function that is isotone, intensive, and idempotent, and its φ -open elements are precisely those $a \in P$ for which $\varphi(a) = a$. Additionally, an element $a \in P$ is called a post-fixed point of φ if $a \leq \varphi(a)$.

A *closure system* on an arbitrary set E is a subset $\mathcal{C}$ of $\langle \mathcal{P}(E), \subseteq \rangle$ that is closed under arbitrary meets and contains E . When φ is a closure on $\mathcal{P}(E)$, its closed elements are called the *closed sets* of φ . An *opening system*, also known as *dual closure system* is defined dually. Hence, an *opening system* on an arbitrary set E is a subset $\mathcal{O}$ of $\mathcal{P}(E)$ that is closed under arbitrary joins and contains $\emptyset$. Similarly as for closure systems, when φ is an opening operator on $\mathcal{P}(E)$, its opened elements are called the *opened sets* of φ . The reader is referred to [6] for a detailed overview of closure and opening operators, as well as closure and opening systems.

2.2 Hesitant fuzzy sets over the power set quantale

In this subsection, we consider a generalized framework for hesitant fuzzy sets in which membership values are taken from the power set quantale of an arbitrary monoid, following the formulation introduced in [24]. This approach extends the well-known definitions introduced and studied by Torra [25] and Pei and Yi [19]. The algebraic properties of quantales are used to establish the main results presented in this paper. Throughout the paper, unless stated otherwise, we denote this quantale by $\mathcal{Q}$.

Let $\mathcal{S} = \langle S, *, 1_* \rangle$ be a monoid and let $Q = \mathcal{P}(S)$ denote its power set. As illustrated in Example 2.1, the algebraic structure $\mathcal{Q} = \langle Q, \cup, \cap, \otimes, \emptyset, S \rangle$, where $\otimes$ is the multiplication defined by (17), is a unital quantale with the multiplicative identity $1_\otimes = \{1_*\}$. This algebraic structure is commonly referred to as the power set quantale of $\mathcal{S}$.

Definition 2.2. ([24]). Let X be a non-empty set (a universal set) and let $\mathcal{S} = \langle S, *, 1_* \rangle$ be a monoid. Let us denote by $Q = \mathcal{P}(S)$ the power set of S and by $\mathcal{Q} = \langle Q, \cup, \cap, \otimes, \emptyset, S \rangle$ the corresponding power set quantale. A *hesitant fuzzy set on X over $\mathcal{Q}$* , or simply a *hesitant fuzzy set on X* , is then defined as any function $h : X \rightarrow \mathcal{Q}$.

Some important examples of hesitant fuzzy sets are:

1. Empty hesitant fuzzy set: $h_\emptyset(x) = \emptyset$, for each $x \in X$;
2. Full hesitant fuzzy set: $h_S(x) = S$, for each $x \in X$.

For a hesitant fuzzy set $h : X \rightarrow \mathcal{Q}$, each $h(x)$ represents the membership value of x in h . The collection of all hesitant fuzzy sets on X with membership values in the quantale $\mathcal{Q}$ is denoted by $\mathcal{Q}^X$.

Let $h_1, h_2 \in \mathcal{Q}^X$. The equality of h_1 and h_2 is defined as the usual equality of functions, i.e., $h_1 = h_2$ if and only if $h_1(x) = h_2(x)$, for all $x \in X$. The inclusion $h_1 \subseteq h_2$ is also defined

pointwise: $h_1 \subseteq h_2$ if and only if $h_1(x) \subseteq h_2(x)$, for all $x \in X$. Endowed with this partial order, the structure $\mathcal{Q}(X) = \langle Q^X, \cup, \cap, h_\emptyset, h_S \rangle$ forms a complete lattice in which $h_\emptyset$ is the least element and h_S is the greatest element, and where the meet (intersection) $\bigcap_{i \in I} h_i$ and the join (union) $\bigcup_{i \in I} h_i$ of an arbitrary family $\{h_i\}_{i \in I}$ of hesitant fuzzy sets on X are functions from X into Q defined by:

$$\left(\bigcap_{i \in I} h_i\right)(x) = \bigcap_{i \in I} h_i(x), \quad \left(\bigcup_{i \in I} h_i\right)(x) = \bigcup_{i \in I} h_i(x),$$

for all $x \in X$. As usual, we identify the structure $\mathcal{Q}(X)$ with the carrier set Q^X . We introduce the *multiplication* of hesitant fuzzy sets as the binary operation $\otimes$ on $\mathcal{Q}(X)$ as in [24]:

$$(h_1 \otimes h_2)(x) = h_1(x) \otimes h_2(x) = \{a * b \mid a \in h_1(x) \text{ and } b \in h_2(x)\}. \quad (31)$$

Clearly, the structure $\langle Q^X, \otimes \rangle$ forms a partially ordered semigroup, meaning that the operation $\otimes$ is associative and isotone with respect to the partial order. In particular, isotonicity implies that for all $h, h_1, h_2 \in Q^X$ and all $x \in X$:

$$h_1 \subseteq h_2 \Rightarrow (h_1 \otimes h)(x) \subseteq (h_2 \otimes h)(x) \quad \text{and} \quad (h \otimes h_1)(x) \subseteq (h \otimes h_2)(x).$$

Furthermore, for any collection of hesitant fuzzy sets $\{h_i\}_{i \in I} \subseteq Q^X$ and any $g \in Q^X$, the following infinite distributivity law holds

$$\left(\bigcup_{i \in I} h_i\right) \otimes g = \bigcup_{i \in I} (h_i \otimes g), \quad g \otimes \left(\bigcup_{i \in I} h_i\right) = \bigcup_{i \in I} (g \otimes h_i). \quad (32)$$

It follows that the structure $\mathcal{Q}(X) = \langle Q^X, \cup, \cap, \otimes, h_\emptyset, h_S \rangle$ forms a quantale. Moreover, $\mathcal{Q}(X)$ is a unital power set quantale with $1_\otimes$ as multiplicative identity.

For arbitrary $h_1, h_2 \in Q^X$, the *right $\otimes$ -residual* $h_2 \bowtie h_1$ of h_1 by h_2 is defined as:

$$(h_2 \bowtie h_1)(x) = h_2(x) \backslash h_1(x) = \{c \in S \mid h_2(x) \otimes c \subseteq h_1(x)\},$$

and the *left $\otimes$ -residual* $h_1 \not\bowtie h_2$ of h_1 by h_2 is defined as follows:

$$(h_1 \not\bowtie h_2)(x) = h_1(x) / h_2(x) = \{c \in S \mid c \otimes h_2(x) \subseteq h_1(x)\},$$

for all $x \in X$. These operations satisfy the residuation property, that is, for all $h, h_1, h_2 \in Q^X$:

$$h_1 \otimes h_2 \subseteq h \Leftrightarrow h_1 \subseteq h \not\bowtie h_2 \Leftrightarrow h_2 \subseteq h_1 \bowtie h. \quad (33)$$

In the case where $\otimes$ is commutative, the left and right residuals coincide and we use the following notation:

$$h_2 \otimes h_1 = h_2 \bowtie h_1 = h_1 \not\bowtie h_2.$$

The operation $\otimes$ is called the *residuum operation*. In this case, the residuation property is expressed as:

$$h_1 \otimes h_2 \subseteq h \Leftrightarrow h_1 \subseteq h_2 \otimes h, \quad (34)$$

for arbitrary hesitant fuzzy sets $h, h_1, h_2 \in Q^X$. The commutativity of $\otimes$ also enables the introduction of the *biresiduum operation* $\bowtie$, which is defined by:

$$(h_1 \bowtie h_2)(x) \equiv (h_1 \otimes h_2)(x) \cap (h_2 \otimes h_1)(x), \quad (35)$$

for all $h_1, h_2 \in Q^X$. In the commutative setting, the *complement* $h^\neg$ of $h \in Q^X$ is a hesitant fuzzy set on X , defined by $h^\neg(x) = \neg(h(x))$ for every $x \in X$.

2.3 Additional properties and operations on hesitant fuzzy sets

In this section, we introduce some notions used throughout the remainder of the paper. Observe that the structure $\mathcal{Q}^* = \langle Q, \cup, \otimes, \emptyset, 1_\otimes \rangle$ forms a semiring reduct of the complete lattice $\mathcal{Q} = \langle Q, \cup, \cap, \emptyset, S \rangle$. For any $\lambda \in Q$ and $h \in Q^X$ we introduce the *left scalar multiplication* $\lambda \otimes h \in Q^X$ and the *right scalar multiplication* $h \otimes \lambda \in Q^X$, respectively, as follows:

$$\begin{aligned} (\lambda \otimes h)(x) &= \lambda \otimes h(x) = \{a * b \mid a \in \lambda \text{ and } b \in h(x)\}, \\ (h \otimes \lambda)(x) &= h(x) \otimes \lambda = \{a * b \mid a \in h(x) \text{ and } b \in \lambda\}, \end{aligned}$$

for all $x \in X$. In the case where multiplication $\otimes$ is commutative, the left and right scalar multiplication coincide, i.e., $\lambda \otimes h = h \otimes \lambda$ for all $x \in X$. Consequently, the monoid $\langle Q^X, \cup, h_\emptyset \rangle$ forms a strong $\mathcal{Q}^*$ -bisemimodule. In this setting, the scalar multiplication refers to either the left or right scalar multiplication, but we use the left notation $(\lambda, h) \mapsto \lambda \otimes h$ to represent it. The lattice $\mathcal{Q}(X)$ with this scalar multiplication is denoted by $\mathcal{Q}_\otimes(X)$ and referred to as the $\mathcal{Q}$ -lattice of hesitant fuzzy sets on X . In addition, we say that a hesitant fuzzy set $h \in Q^X$ is a linear combination of hesitant fuzzy sets $h_i \in Q^X (i \in I)$ if h can be represented as follows:

$$h = \bigcup_{i \in I} \lambda_i \otimes h_i. \quad (36)$$

A family $\{h_i\}_{i \in I} \subseteq Q^X$ is referred to as a *complete $\mathcal{Q}$ -sublattice* of $\mathcal{Q}_\otimes(X)$ if it is closed under scalar multiplication and arbitrary joins and meets.

Let $\lambda \in Q$ be an arbitrary scalar and let $h \in Q^X$. The *right s -residual* of h by λ , denoted by $\lambda \backslash_s h$, and the *left s -residual* of h by λ , denoted by $h /_s \lambda$, are hesitant fuzzy sets on X defined, respectively, for all $x \in X$:

$$(\lambda \backslash_s h)(x) = \lambda \backslash h(x), \quad (h /_s \lambda)(x) = h(x) / \lambda. \quad (37)$$

Functions $h \mapsto \lambda \backslash_s h$ and $h \mapsto h /_s \lambda$ are called respectively the *right scalar residuation* and the *left scalar residuation*, respectively. Clearly, for all $x \in X$, we have:

$$(\lambda \backslash_s h)(x) = \{c \in S \mid \lambda \otimes \{c\} \subseteq h(x)\}, \quad (h /_s \lambda)(x) = \{c \in S \mid \{c\} \otimes \lambda \subseteq h(x)\}. \quad (38)$$

The following residuation properties hold for all $\lambda \in Q$ and $h, h' \in Q^X$:

$$\lambda \otimes h' \subseteq h \Leftrightarrow h' \subseteq \lambda \backslash_s h, \quad h' \otimes \lambda \subseteq h \Leftrightarrow h' \subseteq h /_s \lambda. \quad (39)$$

If $\otimes$ is commutative, the right and left s -residuals coincide and we adopt the following notation:

$$\lambda \xrightarrow{s} h = \lambda \backslash_s h = h /_s \lambda.$$

The following notions are inspired by their counterparts in classical fuzzy set theory (cf. [13]) and are adapted here to the hesitant fuzzy setting. Throughout the remainder of this section, we assume that the underlying monoid S is commutative. Consequently, the power set quantale $\mathcal{Q}$ of S is also commutative and admits a unique residuum operation. Given two hesitant fuzzy sets $h_1, h_2 \in Q^X$, we define the following functions:

$$I(h_1, h_2) = \bigcap_{x \in X} h_1(x) \rightarrow h_2(x), \quad (40)$$

$$E(h_1, h_2) = \bigcap_{x \in X} h_1(x) \leftrightarrow h_2(x), \quad (41)$$

$$O(h_1, h_2) = \bigcup_{x \in X} h_1(x) \otimes h_2(x). \quad (42)$$

The resulting functions $I(h_1, h_2)$, $E(h_1, h_2)$ and $O(h_1, h_2)$ are referred to as the *hesitant implication*, *hesitant biimplication*, and *hesitant multiplication*, respectively. Intuitively, $I(h_1, h_2)$ measures the degree to which h_1 implies h_2 , $E(h_1, h_2)$ quantifies their mutual agreement or similarity, and $O(h_1, h_2)$ represents the mutual effect of h_1 and h_2 under multiplication.

Assume $\mathcal{C}$ is a closure system in $\mathcal{Q}(X)$. Let $\mathcal{H} = \{h_i\}_{i \in I}$ be a family of hesitant fuzzy sets on X . The $\mathcal{C}$ -closure of $\mathcal{H}$ is defined by:

$$\mathcal{C}_{\mathcal{H}} = \bigcap \{g \in \mathcal{C} \mid h_i \subseteq g, \text{ for every } i \in I\}. \quad (43)$$

It is the least element of $\mathcal{C}$ containing all members of $\mathcal{H}$. If $\mathcal{H}$ consists of a single hesitant fuzzy set h , then its $\mathcal{C}$ -closure is denoted by $\mathcal{C}_h$ and referred to as the $\mathcal{C}$ -closure of h . Moreover, it is easy to verify that a $\mathcal{C}$ -closure of $\mathcal{H} = \{h_i\}_{i \in I}$ coincides with the $\mathcal{C}$ -closure $\mathcal{C}_h$ of their union, where $h = \bigcup_{i \in I} h_i$.

Analogously, let $\mathcal{C}$ be an opening system in $\mathcal{Q}(X)$. For a family $\mathcal{H} = \{h_i\}_{i \in I}$ of hesitant fuzzy sets on X , the $\mathcal{C}$ -opening of $\mathcal{H}$ is defined by:

$$\mathcal{C}_{\mathcal{H}} = \bigcup \{g \in \mathcal{C} \mid g \subseteq h_i, \text{ for every } i \in I\}. \quad (44)$$

It is the greatest element of $\mathcal{C}$ contained in all members of $\mathcal{H}$. If $\mathcal{H}$ consists of a single hesitant fuzzy set h on X , its $\mathcal{C}$ -opening is denoted by $\mathcal{C}_h$ and called the $\mathcal{C}$ -opening of h . Furthermore, the $\mathcal{C}$ -opening of $\mathcal{H} = \{h_i\}_{i \in I}$ coincides with the $\mathcal{C}$ -opening $\mathcal{C}_h$ of their intersection, where $h = \bigcap_{i \in I} h_i$.

If $\mathcal{C}$ is a closure system in $\mathcal{Q}(X)$, it necessarily contains the full hesitant fuzzy set h_S on X . Consequently, for every $x \in X$, the collection $\{h \in \mathcal{C} \mid 1_{\otimes} \subseteq h(x)\}$ is non-empty, as it contains at least this maximal element. Hence, we define:

$$\mathcal{C}_x = \bigcap \{h \in \mathcal{C} \mid 1_{\otimes} \subseteq h(x)\} \in \mathcal{C}. \quad (45)$$

The hesitant fuzzy set $\mathcal{C}_x$ is the $\mathcal{C}$ -closure of the singleton crisp set $x \in X$ and is referred to as the *principal element* of $\mathcal{C}$ generated by x . The collection

$$\mathcal{P}(\mathcal{C}) = \{\mathcal{C}_x \mid x \in X\} \quad (46)$$

is called the *principal part* of $\mathcal{C}$.

Given a closure operator φ on the lattice $\mathcal{Q}(X)$, the family of all hesitant fuzzy sets $h \in \mathcal{Q}^X$ that are fixed points of φ , i.e., those satisfying $\varphi(h) = h$, forms a closure system. Conversely, for any closure system $\mathcal{C} \subseteq \mathcal{Q}(X)$, there exists a unique closure operator φ such that $\mathcal{C}$ consists precisely of the φ -closed elements. Similarly, if φ is an opening operator on $\mathcal{Q}(X)$, then the collection of all φ -open elements forms an opening system in $\mathcal{Q}(X)$. Conversely, every opening system in $\mathcal{Q}(X)$ arises as the set of fixed points of a unique opening operator. The concepts related to closures and openings of fuzzy sets are thoroughly discussed by Ignjatović et al. [13].

3 Hesitant fuzzy relations over the power set quantale

Throughout the remainder of the paper, we assume that the underlying monoid S is commutative. Consequently, the power set quantale $\mathcal{Q}$ of S is also commutative and admits a unique residuum operation.

In this section, we introduce hesitant fuzzy relations on the universal set X over the power set quantale $\mathcal{Q}$, following the basic definitions given by Tasić et al. [24]. Formally, a *hesitant fuzzy relation* R on X is defined as any function from $X \times X$ into $\mathcal{Q}$. Equivalently, R can be viewed as a hesitant fuzzy set on $X \times X$.

The standard notions of equality, inclusion, joins, meets and ordering of hesitant fuzzy relations are defined analogously to those for hesitant fuzzy sets. The *composition* of two hesitant fuzzy relations R_1 and R_2 on X is a hesitant fuzzy relation $R_1 \circ R_2$ on X defined by:

$$(R_1 \circ R_2)(x, y) = \bigcup_{a \in X} (R_1(x, a) \otimes R_2(a, y)), \quad (47)$$

for all $x, y \in X$. Let R be a hesitant fuzzy relation on X . The *inverse* of R is the hesitant fuzzy relation R^{-1} on X such that $R^{-1}(y, x) = R(x, y)$ for all $x, y \in X$. Clearly, $(R^{-1})^{-1} = R$. If $\mathcal{Q}$ is a commutative power set quantale, one can easily show that the following identity holds for every $x \in X$:

$$(h \circ R^{-1})(x) = (R \circ h)(x). \quad (48)$$

The *identity hesitant fuzzy relation* Δ_X on X is defined as:

$$\Delta_X(x, y) = \begin{cases} 1_{\otimes} & \text{if } x = y, \\ \emptyset & \text{if } x \neq y, \end{cases} \quad (49)$$

for all $x, y \in X$. The *zero hesitant fuzzy relation* $\mathbf{0}$ and the *full hesitant fuzzy relation* $\mathbf{I}$ on X are defined by $\mathbf{0}(x, y) = \emptyset$ and $\mathbf{I}(x, y) = h_S$, respectively, for all $x, y \in X$. The set of all hesitant fuzzy relations on X is denoted by $Q^{X \times X}$.

For any $h \in Q^X$ and $R \in Q^{X \times X}$, the compositions $h \circ R$ and $R \circ h$ are hesitant fuzzy sets on X defined by:

$$(h \circ R)(x) = \bigcup_{y \in X} h(y) \otimes R(y, x), \quad (R \circ h)(x) = \bigcup_{y \in X} R(x, y) \otimes h(y), \quad (50)$$

for all $x \in X$. The composition $h_1 \circ h_2$ of two hesitant fuzzy sets $h_1, h_2 \in Q^X$ is defined by:

$$(h_1 \circ h_2)(x) = \bigcup_{x \in X} h_1(x) \otimes h_2(x), \quad (51)$$

for all $x \in X$. For a hesitant fuzzy relation R on X we define the *n-th power* R^n of R inductively as follows: $R^0 = \Delta_X$ and $R^{n+1} = R^n \circ R$, for each $n \in \mathbb{N}_0$, where $\mathbb{N}_0$ represents the set of non-negative integers.

Let $R, T \in Q^{X \times X}$, $h, g \in Q^X$ and $\{h_i\}_{i \in I} \subseteq Q^X$. Then the following properties hold:

$$(h \circ R) \circ T = h \circ (R \circ T), \quad (h \circ R) \circ g = h \circ (R \circ g), \quad (R \circ T) \circ h = R \circ (T \circ h), \quad (52)$$

$$R \subseteq T \Rightarrow R \circ h \subseteq T \circ h, \quad h \circ R \subseteq h \circ T, \quad (53)$$

$$h \circ \left(\bigcup_{i \in I} R_i \right) = \bigcup_{i \in I} (h \circ R_i), \quad \left(\bigcup_{i \in I} R_i \right) \circ h = \bigcup_{i \in I} (R_i \circ h), \quad (54)$$

$$h \circ \left(\bigcap_{i \in I} R_i \right) \subseteq \bigcap_{i \in I} (h \circ R_i), \quad \left(\bigcap_{i \in I} R_i \right) \circ h \subseteq \bigcap_{i \in I} (R_i \circ h), \quad (55)$$

$$\left(\bigcup_{i \in I} h_i \right) \circ R = \bigcup_{i \in I} (h_i \circ R), \quad R \circ \left(\bigcup_{i \in I} h_i \right) = \bigcup_{i \in I} (R \circ h_i) \quad (56)$$

$$\left(\bigcap_{i \in I} h_i \right) \circ R \subseteq \bigcap_{i \in I} (h_i \circ R), \quad R \circ \left(\bigcap_{i \in I} h_i \right) \subseteq \bigcap_{i \in I} (R \circ h_i). \quad (57)$$

The algebra $\mathcal{R}(X) = \langle Q^{X \times X}, \cup, \cap, \circ, \mathbf{0}, \mathbf{I} \rangle$ forms a unital quantale in which the least element is the zero hesitant fuzzy relation $\mathbf{0}$, the greatest element is the full hesitant fuzzy relation $\mathbf{I}$, and the identity element is the identity hesitant fuzzy relation Δ_X (cf. [24]). In what follows, we identify the structure $\mathcal{R}(X)$ with its carrier set $Q^{X \times X}$.

For computational purposes, if X is a finite set with n elements, then hesitant fuzzy relations on X are represented as $n \times n$ matrices over Q . Similarly, hesitant fuzzy sets on X are represented as column vectors over Q . For hesitant fuzzy relations $R_1, R_2 \in \mathcal{R}(X)$ and hesitant fuzzy sets $h_1, h_2 \in \mathcal{Q}(X)$, the composition $R_1 \circ R_2$ corresponds to the matrix product. Similarly, $h_1 \circ R_1$ can be viewed as the product of a $1 \times n$ row vector h_1 and an $n \times n$ matrix R_1 , while $R_1 \circ h_1$ corresponds to the product of an $n \times n$ matrix R_1 and an $n \times 1$ column vector h_1^T (the transpose of h_1). Moreover, $h_1 \circ h_2$ can be interpreted as the scalar product of vectors h_1 and h_2 .

Let $R \in Q^{X \times X}$ be a hesitant fuzzy relation on X and let $h \in Q^X$ be a hesitant fuzzy set on X . The *left $\circ$ -residual* of h by R is a hesitant fuzzy set $h \not\circ R$ on X defined by:

$$(h \not\circ R)(x) = \bigcap_{y \in X} R(x, y) \rightarrow h(y), \quad (58)$$

for all $x \in X$, and the *right $\circ$ -residual* of h by R is a hesitant fuzzy set $R \not\circ h$ on X defined by:

$$(R \not\circ h)(x) = \bigcap_{y \in X} R(y, x) \rightarrow h(y), \quad (59)$$

for all $x \in X$. Hence, the following residuation properties hold:

$$h \circ R \subseteq g \Leftrightarrow h \subseteq g \not\circ R, \quad R \circ h \subseteq g \Leftrightarrow h \subseteq R \not\circ g. \quad (60)$$

Observe that, if R is symmetric (see Section 3.1), then $h \not\circ R = R \not\circ h$ for each hesitant fuzzy set h on X .

Additionally, let h and g be hesitant fuzzy sets on X . The *left $\circ$ -residual* of h by g is a hesitant fuzzy relation $h \not\circ g$ on X , defined by:

$$(h \not\circ g)(x, y) = g(y) \rightarrow h(x), \quad (61)$$

for arbitrary $x, y \in X$. The *right $\circ$ -residual* of h by g is a hesitant fuzzy relation $g \not\circ h$ on X , defined by:

$$(g \not\circ h)(x, y) = g(x) \rightarrow h(y), \quad (62)$$

for arbitrary $x, y \in X$. In that case, the following residuation properties hold for arbitrary $h, g \in Q^X$ and $R \in Q^{X \times X}$:

$$g \circ R \subseteq h \Leftrightarrow R \subseteq g \not\circ h, \quad R \circ g \subseteq h \Leftrightarrow R \subseteq h \not\circ g. \quad (63)$$

For arbitrary $h, g \in Q^X$ and $R, T \in Q^{X \times X}$, the following properties can be easily verified:

$$(g \not\circ R) \circ R \subseteq g, \quad R \circ (R \not\circ h) \subseteq h, \quad (64)$$

$$g \not\circ \left(\bigcup_{i \in I} R_i \right) = \bigcap_{i \in I} (g \not\circ R_i), \quad \left(\bigcup_{i \in I} R_i \right) \not\circ g = \bigcap_{i \in I} (R_i \not\circ g). \quad (65)$$

Example 3.1. Let $X = \{x_1, x_2\}$ be a universe and let $\mathcal{Q}$ be a Gödel power set quantale, which is commutative. Consider the hesitant fuzzy relation R on X and the hesitant fuzzy set h on X defined as follows:

$$R = \begin{bmatrix} \{0.1\} & \{0.1, 0.6\} \\ \{0.1, 0.6\} & \{0.2, 1\} \end{bmatrix}, \quad h = \begin{bmatrix} \{0.1, 0.3\} \\ \{0.1, 0.7\} \end{bmatrix}.$$

According to (58), the left residual of h by R is computed as follows:

$$\begin{aligned}
(h \not\vdash R)(x_1) &= (R(x_1, x_1) \rightarrow h(x_1)) \cap (R(x_1, x_2) \rightarrow h(x_2)) \\
&= (\{0.1\} \rightarrow \{0.1, 0.3\}) \cap (\{0.1, 0.6\} \rightarrow \{0.1, 0.7\}) \\
&= \{\gamma \in [0, 1] \mid \gamma \otimes \{0.1\} \subseteq \{0.1, 0.3\}\} \cap \{\gamma \in [0, 1] \mid \gamma \otimes \{0.1, 0.6\} \subseteq \{0.1, 0.7\}\} \\
&= [0.1, 1] \cap \{0.1\} = \{0.1\}, \\
(h \not\vdash R)(x_2) &= (R(x_2, x_1) \rightarrow h(x_1)) \cap (R(x_2, x_2) \rightarrow h(x_2)) \\
&= (\{0.1, 0.6\} \rightarrow \{0.1, 0.3\}) \cap (\{0.2, 1\} \rightarrow \{0.1, 0.7\}) \\
&= \{\gamma \in [0, 1] \mid \gamma \otimes \{0.1, 0.2\} \subseteq \{0.1, 0.3\}\} \cap \{\gamma \in [0, 1] \mid \gamma \otimes \{0.2, 1\} \subseteq \{0.1, 0.7\}\} \\
&= \{0.1\} \cap \{0.1\} = \{0.1\}.
\end{aligned}$$

Similarly, using (59) the right residual of h by R is computed as follows:

$$(R \not\vdash h)(x_1) = (R \not\vdash h)(x_2) = \{0.1\}.$$

Therefore, the left and right residuals of h by R coincide:

$$h \not\vdash R = R \not\vdash h = \begin{bmatrix} \{0.1\} \\ \{0.1\} \end{bmatrix}.$$

3.1 Hesitant fuzzy relations with specific properties

Let R be a hesitant fuzzy relation on X . We employ the standard notions of reflexivity, symmetry, and transitivity, defined in the usual way as in the theory of lattice-valued fuzzy relations.

In particular, a reflexive and transitive hesitant fuzzy relation on X is called a *hesitant fuzzy quasi-order on X* . A reflexive, symmetric and transitive hesitant fuzzy relation on X is called a *hesitant fuzzy equivalence relation on X* (or simply a *hesitant fuzzy equivalence on X*). The collections of all hesitant fuzzy quasi-orders and all hesitant fuzzy equivalences on X are denoted by $\mathcal{F}(X)$ and $\mathcal{E}(X)$, respectively. Moreover, it is well known that every reflexive relation satisfies $R \subseteq R \circ R$, and that reflexivity together with transitivity implies $R \circ R = R$.

Notice that the lattice $\mathcal{R}(X)$ of hesitant fuzzy relations on X can be identified with the lattice of hesitant fuzzy sets on $X \times X$. Consequently, the notions of closure and opening systems in $\mathcal{R}(X)$, as well as the corresponding $\mathcal{C}$ -closure and $\mathcal{C}$ -opening for $\mathcal{C} \subseteq Q^{X \times X}$ are defined analogously to those for hesitant fuzzy sets.

The collection $\mathcal{F}(X)$ of all hesitant fuzzy quasi-orders on X , as well as the collection $\mathcal{E}(X)$ of all hesitant fuzzy equivalence relations on X , form closure systems within the lattice $\mathcal{R}(X)$. The closure of a family of hesitant fuzzy relations (or a single hesitant fuzzy relation) with respect to $\mathcal{F}(X)$ is referred to as the *hesitant fuzzy quasi-order closure*, or simply the $\mathcal{F}$ -closure, while the closure corresponding to $\mathcal{E}(X)$ is known as the *fuzzy equivalence closure*, or $\mathcal{E}$ -closure for short. The collections of all reflexive, symmetric, and transitive fuzzy relations within $\mathcal{R}(X)$ also constitute closure systems.

Accordingly, the *reflexive*, *symmetric*, and *transitive closures* can be defined for a single hesitant fuzzy relation or any family of hesitant fuzzy relations. Let $R \in \mathcal{R}(X)$ be a hesitant fuzzy relation on X . The *reflexive closure* of R is the smallest reflexive hesitant fuzzy relation R^r such that $R \subseteq R^r$, and it can be represented as $R^r = R \cup \Delta_X$. Furthermore, the *symmetric closure* of R is the smallest symmetric hesitant fuzzy relation R^s such that $R \subseteq R^s$, i.e., $R^s = R \cup R^{-1}$. Similarly, the *transitive closure* of R is the smallest transitive hesitant fuzzy relation R^t such that $R \subseteq R^t$ and it holds $R^t = \bigcup_{n \in \mathbb{N}} R^n$. Whenever X is a finite set, the transitive closure algorithm ends after a finite number of steps.

Following the above steps, we can transform an arbitrary hesitant fuzzy relation R into a reflexive, symmetric and transitive hesitant fuzzy relation. The *reflexive-transitive closure* R^f of an arbitrary hesitant fuzzy relation R can be expressed as:

$$R^f = (R^r)^t = (R^t)^r = \bigcup_{n \in \mathbb{N}_0} R^n. \quad (66)$$

Similarly, the *reflexive-symmetric-transitive closure* R^e of R can be expressed as:

$$R^e = ((R^r)^s)^t = ((R^s)^r)^t = ((R^s)^t)^r. \quad (67)$$

Example 3.2. To illustrate the concepts introduced above, we first consider Gödel power set quantale $\mathcal{Q}$ and the hesitant fuzzy relation R on $X = \{x_1, x_2, x_3\}$ with membership values in $\mathcal{Q}$, given by:

$$R = \begin{bmatrix} \{0.1\} & \{0.3\} & \{0.5\} \\ \{1\} & \{0.2\} & \{0.7\} \\ \{0.8\} & \{0.3, 1\} & \{0.5\} \end{bmatrix}.$$

The reflexive closure R^r of R is:

$$R^r = \begin{bmatrix} \{0.1, 1\} & \{0.3\} & \{0.5\} \\ \{1\} & \{0.2, 1\} & \{0.7\} \\ \{0.8\} & \{0.3, 1\} & \{0.5, 1\} \end{bmatrix}.$$

To obtain the reflexive-transitive closure of R , we proceed by computing the transitive closure we will compute the transitive closure of R^r according to the following procedure:

$$(R^r)^2 = \begin{bmatrix} \{0.1, 0.3, 0.5, 1\} & \{0.1, 0.2, 0.3, 0.5\} & \{0.1, 0.3, 0.5\} \\ \{0.1, 0.2, 0.7, 1\} & \{0.2, 0.3, 0.7, 1\} & \{0.2, 0.3, 0.5, 0.7\} \\ \{0.1, 0.3, 0.8, 1\} & \{0.2, 0.3, 0.5, 1\} & \{0.3, 0.5, 0.7, 1\} \end{bmatrix}.$$

Since $((R^r)^2 \circ (R^r)^2)(x_1, x_1) = \{0.1, 0.2, 0.3, 0.5, 1\} \not\subseteq (R^r)^2(x_1, x_1)$, it follows that $(R^r)^2$ is not transitive. Next, we compute $(R^r)^3$:

$$(R^r)^3 = \begin{bmatrix} \{0.1, 0.2, 0.3, 0.5, 1\} & \{0.1, 0.2, 0.3, 0.5\} & \{0.1, 0.2, 0.3, 0.5\} \\ \{0.1, 0.2, 0.3, 0.5, 0.7, 1\} & \{0.1, 0.2, 0.3, 0.5, 0.7, 1\} & \{0.1, 0.2, 0.3, 0.5, 0.7\} \\ \{0.1, 0.2, 0.3, 0.5, 0.8, 1\} & \{0.1, 0.2, 0.3, 0.5, 0.7, 1\} & \{0.1, 0.2, 0.3, 0.5, 0.7, 1\} \end{bmatrix}.$$

Note that $((R^r)^3 \circ (R^r)^3)(x_3, x_1) = \{0.1, 0.2, 0.3, 0.5, 0.7, 0.8, 1\} \not\subseteq (R^r)^3(x_3, x_1)$, hence $(R^r)^3$ is not transitive.

$$(R^r)^4 = \begin{bmatrix} \{0.1, 0.2, 0.3, 0.5, 1\} & \{0.1, 0.2, 0.3, 0.5\} & \{0.1, 0.2, 0.3, 0.5\} \\ \{0.1, 0.2, 0.3, 0.5, 0.7, 1\} & \{0.1, 0.2, 0.3, 0.5, 0.7, 1\} & \{0.1, 0.2, 0.3, 0.5, 0.7\} \\ \{0.1, 0.2, 0.3, 0.5, 0.7, 0.8, 1\} & \{0.1, 0.2, 0.3, 0.5, 0.7, 1\} & \{0.1, 0.2, 0.3, 0.5, 0.7, 1\} \end{bmatrix}.$$

It holds that $(R^r)^4 \circ (R^r)^4 = (R^r)^4$ and $(R^r)^4 = R^r \cup (R^r)^2 \cup (R^r)^3 \cup (R^r)^4$. Consequently, $(R^r)^4$ is the reflexive-transitive closure of R .

In the following, we introduce fundamental concepts associated with hesitant fuzzy relations, which generalize the corresponding notions from classical and fuzzy relational theory and play an important role in the analysis of their structural properties. Let R be a hesitant fuzzy relation on X . For each $x \in X$, the *R-afterset* of x is the hesitant fuzzy set $xR \in \mathcal{Q}(X)$ defined by $xR(y) = R(x, y)$, and the *R-foreset* of x is the hesitant fuzzy set $Rx \in \mathcal{Q}(X)$ defined by $Rx(y) = R(y, x)$, for any $y \in X$. The collection of all *R*-aftersets is denoted by $R^{[a]}$, while the collection of all *R*-foresets is denoted by $R^{[f]}$.

If E is a hesitant fuzzy equivalence relation, we denote E_x instead of $xE = Ex$ and we refer to E_x as the *hesitant fuzzy equivalence class of E* , or just an *E -class*, determined by the element x . Hence, $E_x(a) = E(x, a)$ for all $a \in X$. The collection of all equivalence classes of E , called the factor set of X with respect to E is denoted by $E^{[e]} = \{E_x \mid x \in X\}$.

If X is a finite set containing n elements, then a hesitant fuzzy relation R on X can be represented as an $n \times n$ hesitant fuzzy matrix over $\mathcal{Q}$. In that case, R -aftersets correspond to row vectors and R -foresets correspond to column vectors of this matrix.

Lemma 3.1. Let h be a hesitant fuzzy set on X . The left and right $\circ$ -residuals of h with respect to itself, denoted by $h \not\circ h$ and $h \backslash h$, respectively, are both hesitant fuzzy quasi-orders on X .

Proof. For any hesitant fuzzy set h on X , it is straightforward to verify that $1_{\otimes} \subseteq (h \not\circ h)(x, x)$ and $1_{\otimes} \subseteq (h \backslash h)(x, x)$ for all $x \in X$. Consequently, by property (12), both hesitant fuzzy relations $h \not\circ h$ and $h \backslash h$, defined in (61) and (62) as the left and right $\circ$ -residuals of h with respect to itself, are hesitant fuzzy quasi-orders on X . $\square$

Proposition 3.1. Let F_1 and F_2 be two hesitant fuzzy quasi-orders on X . Then the following equivalence holds:

$$F_1 \subseteq F_2 \quad \Leftrightarrow \quad F_1 \circ F_2 = F_2 \circ F_1 = F_2. \quad (68)$$

Proof. The statement is a direct consequence of standard properties of hesitant fuzzy quasi-orders, hence the proof is omitted. $\square$

Recall that a hesitant fuzzy set $h \in \mathcal{Q}^X$ is called *normalized* if there exists at least one $x \in X$ such that $1_{\otimes} \subseteq h(x)$ [24]. We next state a lemma that will be used in the proof of the following theorem.

Lemma 3.2. [24] Let $h, g \in \mathcal{Q}^X$. If h is normalized, i.e., if there exists $a \in X$ such that $1_{\otimes} \subseteq h(a)$, then, for every $x \in X$ the following properties hold:

$$g(x) \subseteq h(a) \otimes g(x), \quad (69)$$

$$h(a) \rightarrow g(x) \subseteq g(x). \quad (70)$$

Proof. Property (69) follows from the isotonicity of multiplication, while property (70) follows from the antitonicity of the residuum in its first argument. $\square$

Proposition 3.2. Let R be an arbitrary hesitant fuzzy relation on X . The following properties hold:

1. Functions $h \mapsto h \circ R$, $h \mapsto h \not\circ R$, $h \mapsto R \circ h$ and $h \mapsto R \backslash h$ are isotone self-maps of the lattice $\mathcal{Q}(X)$.
2. If R is reflexive, the functions $h \mapsto h \circ R$ and $h \mapsto R \circ h$ are extensive, while $h \mapsto h \not\circ R$ and $h \mapsto R \backslash h$ are intensive.
3. If R is a hesitant fuzzy quasi-order, then each of the functions $h \mapsto h \circ R$, $h \mapsto h \not\circ R$, $h \mapsto R \circ h$ and $h \mapsto R \backslash h$ is idempotent.

Proof. 1. Let h_1 and h_2 be hesitant fuzzy sets on X such that $h_1 \subseteq h_2$. First, we show that $h \mapsto h \circ R$ is isotone. By the isotonicity of $\otimes$, the following holds for all $x \in X$:

$$h_1 \circ R(x) = \bigcup_{y \in X} h_1(y) \otimes R(y, x) \subseteq \bigcup_{y \in X} h_2(y) \otimes R(y, x) = h_2 \circ R(x).$$

Hence, $h \mapsto h \circ R$ is isotone. Next, we consider $h \mapsto h \not\circ R$. Since the left $\circ$ -residual $\not\circ$ is isotone in its second argument and $h_1 \subseteq h_2$, it follows that for all $x \in X$:

$$h_1 \not\circ R(x) = \bigcap_{y \in X} R(x, y) \rightarrow h_1(y) \subseteq \bigcap_{y \in X} R(x, y) \rightarrow h_2(y) = h_2 \not\circ R(x).$$

Therefore, $h \mapsto h \not\circ R$ is isotone. The isotonicity of the functions $h \mapsto R \circ h$ and $h \mapsto R \not\circ h$ follows by analogous reasoning.

2. Let R be a reflexive hesitant fuzzy relation on X , that is, $1_\otimes \subseteq R(x, x)$ for all $x \in X$. We first show that $h \mapsto h \circ R$ is extensive. By reflexivity, for every $x \in X$ we have:

$$h(x) \subseteq h(x) \otimes R(x, x) \subseteq \bigcup_{y \in X} h(y) \otimes R(y, x) = h \circ R(x).$$

Hence, $h \mapsto h \circ R$ is extensive. Next, we consider the mapping $h \mapsto h \not\circ R$. Since $1_\otimes \subseteq R(x, x)$, Lemma 3.2 implies that for every hesitant fuzzy set h on X the following holds:

$$h \not\circ R(x) = \bigcap_{y \in X} R(x, y) \rightarrow h(y) \subseteq R(x, x) \rightarrow h(x) \subseteq h(x),$$

for all $x \in X$. Thus, $h \mapsto h \not\circ R$ is intensive. The remaining assertions follow by analogous arguments.

3. Assume that R is a hesitant fuzzy quasi-order. First, we show that $h \mapsto h \circ R$ is idempotent, that is, $(h \circ R) \circ R = h \circ R$. Since $h \mapsto h \circ R$ is extensive, we immediately have $(h \circ R) \circ R \supseteq h \circ R$. For the reverse inclusion, we use the transitivity of R and the isotonicity of $\otimes$. For all $x \in X$ we have:

$$\begin{aligned} ((h \circ R) \circ R)(x) &= \bigcup_{y \in X} \left((h \circ R)(y) \otimes R(y, x) \right) = \bigcup_{y \in X} \left(\left(\bigcup_{z \in X} h(z) \otimes R(z, y) \right) \otimes R(y, x) \right) \\ &= \bigcup_{y \in X} \bigcup_{z \in X} \left(h(z) \otimes R(z, y) \otimes R(y, x) \right) \subseteq \bigcup_{z \in X} \left(h(z) \otimes R(z, x) \right) = (h \circ R)(x). \end{aligned}$$

Thus, $((h \circ R) \circ R) = (h \circ R)$ for every $h \in Q^X$ and the function $h \mapsto h \circ R$ is idempotent. Next, we show the idempotency of $h \mapsto h \not\circ R$. Since it is intensive, we immediately obtain $(h \not\circ R) \supseteq ((h \not\circ R) \not\circ R)$. For the converse, recall that the transitivity of R yields $R(x, y) \otimes R(y, z) \subseteq R(x, z)$, for all $x, y, z \in X$. By the antitonicity of the left residuum in its first argument, this implies:

$$(R(x, y) \otimes R(y, z)) \rightarrow h(z) \supseteq R(x, z) \rightarrow h(z),$$

for all $x, y, z \in X$. By properties (13) and (9), together with the transitivity of R and antitonicity of the residuum in its first argument, we obtain:

$$\begin{aligned} ((h \not\circ R) \not\circ R)(x) &= \bigcap_{y \in X} \left(R(x, y) \rightarrow \bigcap_{z \in X} (R(y, z) \rightarrow h(z)) \right) = \bigcap_{y \in X} \bigcap_{z \in X} ((R(x, y) \otimes R(y, z)) \rightarrow h(z)) \\ &\supseteq \bigcap_{y \in X} \bigcap_{z \in X} \left(R(x, z) \rightarrow h(z) \right) = \bigcap_{y \in X} (h \not\circ R)(x) = (h \not\circ R)(x), \end{aligned}$$

for all $x \in X$. Therefore, $(h \not\circ R) \not\circ R = h \not\circ R$, and the function $h \mapsto h \not\circ R$ is idempotent. The remaining assertions follow by similar arguments. $\square$

4 Hesitant fuzzy relation inequalities

This paper focuses on specific classes of hesitant fuzzy relation inequalities, which are formally introduced and analyzed in this section. Throughout the section, R denotes a given hesitant fuzzy relation on X , while w denotes an *unknown* taking values in Q^X .

Here we consider *hesitant fuzzy relation inequalities* on X of the form

$$w \circ R \subseteq w, \quad R \circ w \subseteq w, \quad w \subseteq w \not\prec R, \quad w \subseteq R \not\prec w, \quad (71)$$

as well as *hesitant fuzzy relation equations* on X of the form

$$w \circ R = w, \quad R \circ w = w, \quad w = w \not\prec R, \quad w = R \not\prec w. \quad (72)$$

Let Φ denote an arbitrary hesitant fuzzy relation inequality or equation of the form (71) or (72). Then the set of all solutions to Φ is denoted by $\langle \Phi \rangle$.

A family $\{\Phi_i\}_{i \in I}$ of hesitant fuzzy relation inequalities (resp. equations) is referred to as a *system of hesitant fuzzy relation inequalities* (resp. *equations*). A hesitant fuzzy set $h \in Q^X$ is a solution to the system $\{\Phi_i\}_{i \in I}$ if it is a solution to each hesitant fuzzy relation inequality or equation from the system. The set of all solutions to the system $\{\Phi_i\}_{i \in I}$ is denoted by $\langle \Phi_i \mid i \in I \rangle$. In particular, for a finite set $I = \{1, \dots, n\}$, we write $\langle \Phi_1, \dots, \Phi_n \rangle$ instead of $\langle \Phi_i \mid i \in I \rangle$.

Theorem 4.1. *Let $\{R_i\}_{i \in I}$ be a family of hesitant fuzzy relations on X , let $R = \bigcup_{i \in I} R_i$ and $F = R^f$. Then*

$$\begin{aligned} \langle w \circ R_i \subseteq w \mid i \in I \rangle &= \langle w \circ R \subseteq w \rangle = \langle w \circ F \subseteq w \rangle = \langle w \circ F = w \rangle \\ &= \langle w \subseteq w \not\prec R_i \mid i \in I \rangle = \langle w \subseteq w \not\prec R \rangle = \langle w \subseteq w \not\prec F \rangle = \langle w = w \not\prec F \rangle. \end{aligned} \quad (73)$$

In addition, these sets consist of all hesitant fuzzy sets on X that are linear combinations of F -aftersets.

Proof. Due to the residuation property (60), the following equalities hold:

$$\begin{aligned} \langle w \circ R_i \subseteq w \mid i \in I \rangle &= \langle w \subseteq w \not\prec R_i \mid i \in I \rangle, \quad \langle w \circ R \subseteq w \rangle = \langle w \subseteq w \not\prec R \rangle, \\ \langle w \circ F \subseteq w \rangle &= \langle w \subseteq w \not\prec F \rangle. \end{aligned}$$

Let $h \in \langle w \circ R_i \subseteq w \mid i \in I \rangle$, i.e., $h \circ R_i \subseteq h$. Then, by property (54), we obtain:

$$h \circ R = h \circ \left(\bigcup_{i \in I} R_i \right) = \bigcup_{i \in I} (h \circ R_i) \subseteq h.$$

Hence, $\langle w \circ R_i \subseteq w \mid i \in I \rangle \subseteq \langle w \circ R \subseteq w \rangle$. Conversely, let $h \in \langle w \circ R \subseteq w \rangle$, that is, assume that $h \circ R \subseteq h$. Since $R_i \subseteq R$ for all $i \in I$, property (53) implies that $h \circ R_i \subseteq h \circ R \subseteq h$ for all $i \in I$. Consequently, we obtain $\langle w \circ R \subseteq w \rangle \subseteq \langle w \circ R_i \subseteq w \mid i \in I \rangle$. Therefore, the equality $\langle w \circ R_i \subseteq w \mid i \in I \rangle = \langle w \circ R \subseteq w \rangle$ holds.

Next, let $h \in \langle w \circ R \subseteq w \rangle$. Then, since $h \circ R^0 \subseteq h$, it follows by induction that $h \circ R^n \subseteq h$ for all $n \in \mathbb{N}_0$. Consequently, the following holds:

$$h \circ F = h \circ \left(\bigcup_{n \in \mathbb{N}_0} R^n \right) = \bigcup_{n \in \mathbb{N}_0} (h \circ R^n) \subseteq h.$$

Hence, $\langle w \circ R \subseteq w \rangle \subseteq \langle w \circ F \subseteq w \rangle$. Conversely, suppose that $h \in \langle w \circ F \subseteq w \rangle$, that is, $h \circ F \subseteq h$. If $R \subseteq F$, it follows that $h \circ R \subseteq h \circ F \subseteq h$, and, consequently, $\langle w \circ F \subseteq w \rangle \subseteq \langle w \circ R \subseteq w \rangle$. Therefore, the equality $\langle w \circ R \subseteq w \rangle = \langle w \circ F \subseteq w \rangle$ holds.

Assume now $h \in \langle w \circ F \subseteq w \rangle$. For any $h \in Q^X$, the reflexivity of F implies that $h \subseteq h \circ F$. Together with $h \circ F \subseteq h$, this yields $h \circ F = h$, and, consequently, $\langle w \circ F \subseteq w \rangle \subseteq \langle w \circ F = w \rangle$. The opposite implication is evident, so we obtain $\langle w \circ F \subseteq w \rangle = \langle w \circ F = w \rangle$. Similarly, since $h \mapsto h \not\subseteq F$ is intensive (Proposition 3.2), we have $\langle w \subseteq w \not\subseteq F \rangle = \langle w = w \not\subseteq F \rangle$. Thus, all the equations in (73) have been proved.

We now show that the set $\langle w \circ F = w \rangle$ consists of all hesitant fuzzy sets on X that can be represented as a linear combination of F -aftersets. For any $h \in \langle w \circ F = w \rangle$ and any $y \in X$, we have:

$$h(y) = (h \circ F)(y) = \bigcup_{x \in X} h(x) \otimes F(x, y) = \bigcup_{x \in X} h(x) \otimes xF(y) = \left(\bigcup_{x \in X} h(x) \otimes (xF) \right)(y).$$

Consequently:

$$h = \bigcup_{x \in X} h(x) \otimes (xF) = \bigcup_{x \in X} \lambda_x \otimes (xF),$$

where $\lambda_x = h(x)$, for each $x \in X$. Hence, h is represented as a linear combination of F -aftersets. Conversely, assume that h can be represented as a linear combination of F -aftersets, i.e., there are scalars $\lambda_x \in Q$, $x \in X$, such that:

$$h = \bigcup_{x \in X} \lambda_x \otimes (xF).$$

Then it follows:

$$\begin{aligned} (h \circ F)(y) &= \bigcup_{z \in X} h(z) \otimes F(z, y) = \bigcup_{z \in X} \left(\bigcup_{x \in X} \lambda_x \otimes (xF) \right)(z) \otimes F(z, y) \\ &= \bigcup_{z \in X} \left(\bigcup_{x \in X} \lambda_x \otimes F(x, z) \right) \otimes F(z, y) = \bigcup_{x \in X} \lambda_x \otimes \left(\bigcup_{z \in X} F(x, z) \otimes F(z, y) \right) \\ &= \bigcup_{x \in X} \lambda_x \otimes F(x, y) = \left(\bigcup_{x \in X} \lambda_x \otimes xF \right)(y) = h(y), \end{aligned}$$

for any $y \in X$. Therefore, $h \in \langle w \circ F = w \rangle$, which completes the proof. $\square$

Using an analogous approach, we can show the following theorem.

Theorem 4.2. *Let $\{R_i\}_{i \in I}$ be a family of hesitant fuzzy relations on X , let $R = \bigcup_{i \in I} R_i$ and $F = R^f$. Then*

$$\begin{aligned} \langle R_i \circ w \subseteq w \mid i \in I \rangle &= \langle R \circ w \subseteq w \rangle = \langle F \circ w \subseteq w \rangle = \langle F \circ w = w \rangle \\ &= \langle w \subseteq R_i \not\subseteq w \mid i \in I \rangle = \langle w \subseteq R \not\subseteq w \rangle = \langle w \subseteq F \not\subseteq w \rangle = \langle w = F \not\subseteq w \rangle. \end{aligned} \quad (74)$$

In addition, these sets consist of all hesitant fuzzy sets on X that are linear combinations of F -foresets.

The following result addresses hesitant fuzzy relation inequalities that involve the equivalence closure of the corresponding hesitant fuzzy relation.

Theorem 4.3. *Let $\{R_i\}_{i \in I}$ be a family of hesitant fuzzy relations on X , let $R = \bigcup_{i \in I} R_i$ and $E = R^e$. Then*

$$\begin{aligned} \langle w \circ R_i \subseteq w, R_i \circ w \subseteq w \mid i \in I \rangle &= \langle w \circ R \subseteq w, R \circ w \subseteq w \rangle = \langle w \circ E \subseteq w \rangle = \langle w \circ E = w \rangle \\ &= \langle w \subseteq w \not\subseteq R_i, w \subseteq R_i \not\subseteq w \mid i \in I \rangle = \langle w \subseteq w \not\subseteq R, w \subseteq R \not\subseteq w \rangle = \langle w \subseteq w \not\subseteq E \rangle = \langle w = w \not\subseteq E \rangle. \end{aligned} \quad (75)$$

In addition, these sets consist of all hesitant fuzzy sets on A that are linear combinations of E -classes.

Proof. The following equalities are direct consequences of Theorem 4.1 and Theorem 4.2:

$$\begin{aligned} \langle w \circ R_i \subseteq w, R_i \circ w \subseteq w \mid i \in I \rangle &= \langle w \circ R \subseteq w, R \circ w \subseteq w \rangle = \langle w \subseteq w \not\circ R_i, w \subseteq R_i \not\circ w \mid i \in I \rangle \\ &= \langle w \subseteq w \not\circ R, w \subseteq R \not\circ w \rangle. \end{aligned}$$

It follows immediately that $\langle w \circ E \subseteq w \rangle = \langle w \subseteq w \not\circ E \rangle$.

Assume now that $h \in \langle w \circ E \subseteq w \rangle$, i.e., $h \circ E \subseteq h$. Since $h \mapsto h \circ E$ is extensive, we immediately obtain $h \subseteq h \circ E$, and hence $h \circ E = h$. Conversely, if $h \in \langle w \circ E \subseteq w \rangle$ then clearly $h \circ E \subseteq h$. Therefore, $\langle w \circ E \subseteq w \rangle = \langle w \circ E = w \rangle$.

The equality $\langle w \subseteq w \not\circ E \rangle = \langle w = w \not\circ E \rangle$ can be proved in a similar way.

Next, let $h \in \langle w \circ E \subseteq w \rangle$. Since $R \subseteq E$, we obtain $h \circ R \subseteq h \circ E \subseteq h$. Hence, $\langle w \circ E \subseteq w \rangle \subseteq \langle w \circ R \subseteq w, R \circ w \subseteq w \rangle$. Conversely, assume that $h \in \langle w \circ R \subseteq w, R \circ w \subseteq w \rangle$. By properties (54) and (48), we obtain:

$$h \circ R^s = h \circ (R \cup R^{-1}) = (h \circ R) \cup (h \circ R^{-1}) = (h \circ R) \cup (R \circ h) \subseteq h.$$

By induction, $h \circ (R^s)^n \subseteq h$ holds for each $n \in \mathbb{N}_0$. Therefore:

$$h \circ E = h \circ (R^s)^f = h \circ \bigcup_{n \in \mathbb{N}_0} (R^s)^n = \bigcup_{n \in \mathbb{N}_0} h \circ (R^s)^n \subseteq h,$$

which implies $\langle w \circ R \subseteq w, R \circ w \subseteq w \rangle \subseteq \langle w \circ E \subseteq w \rangle$. Hence, $\langle w \circ R \subseteq w, R \circ w \subseteq w \rangle = \langle w \circ E = w \rangle$.

Using the same approach as in the proof of Theorem 4.1, we obtain that $\langle w \circ E = w \rangle$ coincides with the family of all hesitant fuzzy sets on X that can be expressed as linear combinations of E -classes. $\square$

In view of Theorems 4.1-4.3, it suffices to consider the equations $w \circ F = F$, $F \circ w = F$ and $w \circ E = w$ instead of the systems of inequalities (73), (74) and the system of equations (75), respectively.

Theorem 4.4. *Let $\{R_i\}_{i \in I}$ be a family of hesitant fuzzy relations on X . Then the following holds:*

1. *if $h \in \langle w \circ R_i \subseteq w \mid i \in I \rangle$, then $h^- \in \langle R_i \circ w \subseteq w \mid i \in I \rangle$,*
2. *if $h \in \langle R_i \circ w \subseteq w \mid i \in I \rangle$, then $h^- \in \langle w \circ R_i \subseteq w \mid i \in I \rangle$,*
3. *if $h \in \langle w \circ R_i \subseteq w, R_i \circ w \subseteq w \mid i \in I \rangle$, then $h^- \in \langle w \circ R_i \subseteq w, R_i \circ w \subseteq w \mid i \in I \rangle$.*

Proof. 1. Let h be a hesitant fuzzy set on X such that $h \in \langle w \circ R_i \subseteq w \mid i \in I \rangle$ and let $R = \bigcup_{i \in I} R_i$. By Theorem 4.1, it follows that $h \in \langle w \circ R \subseteq w \rangle$. Using properties (14) and (6), together with the commutativity of $\otimes$, we obtain:

$$\begin{aligned} (h \circ R)^-(x) &= (h \circ R)(x) \rightarrow \emptyset = \left(\bigcup_{y \in X} \left(h(y) \otimes R(y, x) \right) \right) \rightarrow \emptyset = \bigcap_{y \in X} \left((h(y) \otimes R(y, x)) \rightarrow \emptyset \right) \\ &= \bigcap_{y \in X} \left((R(y, x) \otimes h(y)) \rightarrow \emptyset \right) = \bigcap_{y \in X} \left(R(y, x) \rightarrow (h(y) \rightarrow \emptyset) \right) \\ &= \bigcap_{y \in X} \left(R(y, x) \rightarrow h^-(y) \right) = (R \not\circ h^-(x)) \end{aligned}$$

for all $x \in X$. Therefore, the equality $(h \circ R)^\neg = R \setminus h^\neg$ holds. Next, by properties (12) and (60) we obtain:

$$h \circ R \subseteq h \Rightarrow h^\neg \subseteq (h \circ R)^\neg \Leftrightarrow h^\neg \subseteq R \setminus h^\neg \Leftrightarrow R \circ h^\neg \subseteq h^\neg.$$

Consequently, $h \in \langle w \circ R \subseteq w \rangle$ implies $h^\neg \in \langle R \circ w \subseteq w \rangle$, and hence, by Theorem 4.2, $h^\neg \in \langle R_i \circ w \subseteq w \mid i \in I \rangle$.

The assertion 2. can be proved using the similar approach, while 3. directly follows from 1. and 2. $\square$

Let R be a hesitant fuzzy relation on X and let h be a hesitant fuzzy set on X . For the purposes of this paper, we define left and right R -closed hesitant fuzzy sets. We say that h is *left R -closed* if, for all $x, y \in X$, $h(x) \otimes R(x, y) \subseteq h(y)$, which is equivalent to $h \circ R \subseteq h$. Similarly, h is *right R -closed* if $R(x, y) \otimes h(y) \subseteq h(x)$, for all $x, y \in X$, or, equivalently, $R \circ h \subseteq h$. A hesitant fuzzy set that is both left and right R -closed is termed *double R -closed*. Note that, if R is symmetric, then $h \circ R = R \circ h$ and left and right R -closedness coincide; in this case we simply speak of *R -closed* hesitant fuzzy sets. The families of left, right, and double R -closed hesitant fuzzy sets on X are denoted by $\Gamma_l(R)$, $\Gamma_r(R)$, and $\Gamma_d(R)$, respectively. Moreover, following the standard fuzzy set theory terminology (cf. [7, 15]), if E is a hesitant fuzzy equivalence, then *E -closed* hesitant fuzzy sets are said to be *extensional with respect to E* .

Motivated by the work of Lai and Zhang [17] we formulate and prove the following lemma, which will be used to prove the next theorem.

Lemma 4.1. Let R be a hesitant fuzzy relation on X and let $\Gamma_l(R)$ denote the family of left R -closed hesitant fuzzy sets on X . Then, for any family $\{h_i\}_{i \in I} \subseteq \Gamma_l(R)$ of hesitant fuzzy sets on X , any hesitant fuzzy set $g \in \Gamma_l(R)$ on X , and any scalar $\lambda \in Q$, the following conditions hold:

1. $h_\emptyset, h_S \in \Gamma_l(R)$,
2. $\bigcup_{i \in I} \{h_i\} \in \Gamma_l(R)$,
3. $\bigcap_{i \in I} \{h_i\} \in \Gamma_l(R)$,
4. $\lambda \otimes g \in \Gamma_l(R)$,
5. $\lambda \xrightarrow{s} g \in \Gamma_l(R)$.

Proof. 1. Clearly, $h_\emptyset \otimes F(x, y) = h_\emptyset$ and $h_S \otimes R(x, y) \subseteq h_S$ for all $x, y \in X$, which implies $h_\emptyset, h_S \in \Gamma_l(R)$.

2. Given that $\{h_i\}_{i \in I} \subseteq \Gamma_l(R)$, we have:

$$h_i(x) \otimes R(x, y) \subseteq h_i(y),$$

for all $i \in I$ and $x, y \in X$. This implies that, for all $x, y \in X$:

$$\left(\bigcup_{i \in I} h_i(x) \right) \otimes R(x, y) = \bigcup_{i \in I} \left(h_i(x) \otimes R(x, y) \right) \subseteq \bigcup_{i \in I} h_i(y).$$

Consequently, $\bigcup_{i \in I} h_i \in \Gamma_l(R)$.

3. The statement can be proved in the same way as statement 2.

4. Given that $g \in \Gamma_l(R)$, we have:

$$g(x) \otimes R(x, y) \subseteq g(y),$$

for all $x, y \in X$. Next, according to property (25) of scalar multiplication, for any $\lambda \in Q$ the following holds:

$$(\lambda \otimes g(x)) \otimes R(x, y) = \lambda \otimes (g(x) \otimes R(x, y)) \subseteq \lambda \otimes g(y),$$

for all $x, y \in X$. Therefore, $\lambda \otimes g \in \Gamma_l(R)$.

5. Given that $g \in \Gamma_l(R)$, the following holds:

$$g(x) \otimes R(x, y) \subseteq g(y) \quad \Leftrightarrow \quad R(x, y) \subseteq g(x) \rightarrow g(y),$$

for all $x, y \in X$. According to property (14), we have:

$$(\lambda \rightarrow g(x)) \otimes (g(x) \rightarrow g(y)) \subseteq (\lambda \rightarrow g(y)),$$

for all $x, y \in X$. Then, by the residuation property (34), we obtain:

$$(g(x) \rightarrow g(y)) \subseteq (\lambda \rightarrow g(x)) \rightarrow (\lambda \rightarrow g(y)),$$

for all $x, y \in X$. Hence, the following holds:

$$R(x, y) \subseteq g(x) \rightarrow g(y) \subseteq (\lambda \rightarrow g(x)) \rightarrow (\lambda \rightarrow g(y)),$$

for all $x, y \in X$. Therefore:

$$(\lambda \rightarrow g(x)) \otimes R(x, y) \subseteq (\lambda \rightarrow g(y)),$$

for all $x, y \in X$. Consequently, $\lambda \xrightarrow{s} g \in \Gamma_l(R)$. $\square$

Remark 4.1. The conditions 1-5 established for $\Gamma_l(R)$ also hold for $\Gamma_r(R)$ and $\Gamma_d(R)$.

The following theorem is devoted to a detailed characterization of the solutions to the considered inequalities using closures and aftersets (foresets).

Theorem 4.5. Let $\{R_i\}_{i \in I}$ be a family of hesitant fuzzy relations on X . Define $R = \bigcup_{i \in I} R_i$, $F = R^f$ and $E = R^e$. Then:

1. $\langle w \circ R_i \subseteq w \mid i \in I \rangle$, $\langle R_i \circ w \subseteq R_i \mid i \in I \rangle$, $\langle w \circ R_i \subseteq w, R_i \circ w \subseteq w \mid i \in I \rangle$ form both closure and opening systems in $\mathcal{Q}(X)$.
2. The principal part of $\langle w \circ R_i \subseteq w \mid i \in I \rangle$ consists of all F -aftersets.
3. The principal part of $\langle R_i \circ w \subseteq w \mid i \in I \rangle$ consists of all F -foresets.
4. The principal part of $\langle w \circ R_i \subseteq w, R_i \circ w \subseteq w \mid i \in I \rangle$ consists of all E -classes.

Proof. 1. The assertion follows directly from Lemma 4.1.

2. To simplify the notation, set $\mathcal{C} = \langle w \circ R_i \subseteq w \mid i \in I \rangle$. By Theorem 4.1, we have $\mathcal{C} = \langle w \circ F = w \rangle$. Let $\mathcal{P}(\mathcal{C}) = \{C_x \mid x \in X\}$ denote the principal part of $\mathcal{C}$. According to definition (45), $C_x = \bigcap \{g \in \mathcal{C} \mid 1_\otimes \subseteq g(x)\}$ for each $x \in X$. Fix an arbitrary $x \in X$ and let xF denote the F -afterset of x . Then, using the reflexivity and transitivity of F , we obtain:

$$\left((xF) \circ F\right)(y) = \bigcup_{z \in X} (xF)(z) \otimes F(z, y) = \bigcup_{z \in X} F(x, z) \otimes F(z, y) = (F \circ F)(x, y) = xF(y),$$

for all $y \in X$. Hence, $xF \in \mathcal{C}$. Next, we show that $C_x = xF$ for all $x \in X$. Due to the reflexivity of F we have $1_\otimes \subseteq xF(x)$ for all $x \in X$, and, consequently, $xF \in \{g \in \mathcal{C} \mid 1_\otimes \subseteq g\}$. Therefore:

$$C_x = \bigcap \{g \in \mathcal{C} \mid 1_\otimes \subseteq g(x)\} \subseteq xF,$$

for all $x \in X$. Consider now an arbitrary $g \in \mathcal{C}$ such that $1_\otimes \subseteq g(x)$ for some $x \in X$. Then, the following holds:

$$(xF)(y) = F(x, y) \subseteq g(x) \otimes F(x, y) \subseteq \bigcup_{z \in X} g(z) \otimes F(z, y) = (g \circ F)(y) = g(y),$$

for all $y \in X$. Hence, we obtain $xF \subseteq g$ and it follows that:

$$xF \subseteq \bigcap \{g \in \mathcal{C} \mid 1_\otimes \subseteq g(x)\} = \mathcal{C}_x,$$

for all $x \in X$. Therefore, $\mathcal{C}_x = xF$ for all $x \in X$ and $\mathcal{P}(\mathcal{C}) = \{\mathcal{C}_x \mid x \in X\} = \{xF \mid x \in X\}$.

3. According to Theorem 4.2 it holds that $\langle R_i \circ w \subseteq w \mid i \in I \rangle = \langle F \circ w = w \rangle$. As in the previous case, we introduce the notation $\mathcal{C} = \langle R_i \circ w \subseteq w \mid i \in I \rangle = \langle F \circ w = w \rangle$ to simplify expressions. The principal part of $\mathcal{C}$ is denoted by $\mathcal{P}(\mathcal{C})$, as usual. Consider an arbitrary $x \in X$. For each $y \in X$ the following holds:

$$(F \circ (Fx))(y) = \bigcup_{z \in X} F(y, z) \otimes (Fx)(z) = \bigcup_{z \in X} F(y, z) \otimes F(z, x) = F \circ F(y, x) = Fx(y).$$

Therefore, $Fx \in \mathcal{C}$. Next, we show that $\mathcal{C}_x = Fx$ for all $x \in X$. Since F is reflexive, then $1_\otimes \subseteq Fx(x)$ for all $x \in X$ and, consequently, we obtain:

$$\mathcal{C}_x = \bigcap \{g \in \mathcal{C} \mid 1_\otimes \subseteq g(x)\} \subseteq Fx.$$

To prove the opposite inclusion, consider an arbitrary $g \in \mathcal{C}$ such that $1_\otimes \subseteq g(x)$. Then for all $y \in X$ the following holds:

$$(Fx)(y) = F(y, x) \subseteq F(y, x) \otimes g(x) \subseteq \bigcup_{z \in X} F(y, z) \otimes g(z) = (F \circ g)(y) = g(y).$$

It follows that $Fx \subseteq g$, i.e., $Fx \subseteq \mathcal{C}_x$, for all $x \in X$. Consequently, $\mathcal{C}_x = Fx$ for all $x \in X$.

4. According to Theorem 4.3 it holds that $\langle w \circ R_i \subseteq w, R_i \circ w \subseteq w \mid i \in I \rangle = \langle w \circ E = w \rangle$. Let $x \in X$ be an arbitrary element and let $\mathcal{C} = \langle w \circ R_i \subseteq w, R_i \circ w \subseteq w \mid i \in I \rangle = \langle w \circ E = w \rangle$. Then, for all $y \in X$ the following holds:

$$(E_x \circ E)(y) = \bigcup_{z \in X} E_x(z) \otimes E(z, y) = \bigcup_{z \in X} E(x, z) \otimes E(z, y) = E \circ E(x, y) = E(x, y) = E_x(y).$$

Consequently, $E_x \in \mathcal{C}$. Since E is reflexive, then $1_\otimes \subseteq E_x(x)$. By definition,

$$\mathcal{C}_x = \bigcap \{g \in \mathcal{C} \mid 1_\otimes \subseteq g(x)\},$$

and, hence, $\mathcal{C}_x \subseteq E_x$ holds for all $x \in X$.

To prove the opposite inclusion, assume that $g \in \mathcal{C}$ satisfies $1_\otimes \subseteq g(x)$. Then, for all $y \in X$ we have:

$$E_x(y) = E(x, y) \subseteq g(x) \otimes E(x, y) \subseteq \bigcup_{z \in X} g(z) \otimes E(z, y) = (g \circ E)(y) = g(y).$$

Therefore, $E_x \subseteq g$, i.e., $E_x \subseteq \mathcal{C}_x$ for all $x \in X$. Finally, we obtain $\mathcal{C}_x = E_x$ for all $x \in X$. $\square$

Consider a family of hesitant fuzzy relations $\{R_i\}_{i \in I}$ on X and define $R = \bigcup_{i \in I} R_i$, $F = R^f$ and $E = R^e$. Next, we define mappings $g \mapsto g^{lc}$, $g \mapsto g^{rc}$, and $g \mapsto g^{dc}$ to denote closure operators on $\mathcal{Q}(X)$ that correspond to the following closure systems, respectively:

$$\langle w \circ R_i \subseteq w \mid i \in I \rangle, \quad \langle R_i \circ w \subseteq w \mid i \in I \rangle \quad \text{and} \quad \langle w \circ R_i \subseteq w, R_i \circ w \subseteq w \mid i \in I \rangle. \quad (76)$$

Analogously, the mappings $g \mapsto g^{lo}$, $g \mapsto g^{ro}$, and $g \mapsto g^{do}$ denote the corresponding opening operators. That is, $g \mapsto g^{lc}$, $g \mapsto g^{rc}$, and $g \mapsto g^{dc}$ represent the least solutions to the systems described in (76), respectively, that include the hesitant fuzzy set g , while $g \mapsto g^{lo}$, $g \mapsto g^{ro}$, and $g \mapsto g^{do}$ correspond to the greatest solutions to the same systems that are contained within g . The following proposition provides a method for computing the least and greatest solutions described above.

Proposition 4.1. Let $\{R_i\}_{i \in I}$ be a family of hesitant fuzzy relations on X , let $R = \bigcup_{i \in I} R_i$, $F = R^f$ and $E = R^e$ and let g be an arbitrary hesitant fuzzy set on X . Then:

$$g^{lc} = g \circ F, \quad g^{rc} = F \circ g, \quad g^{dc} = g \circ E, \quad g^{lo} = g / F, \quad g^{ro} = F \setminus g, \quad g^{do} = g / E. \quad (77)$$

Proof. According to Proposition 3.2, the functions $g \mapsto g \circ F$, $g \mapsto F \circ g$, and $g \mapsto g \circ E$ are closure operators on $\mathcal{Q}(X)$, while $g \mapsto g \not\circ F$, $g \mapsto F \setminus g$, and $g \mapsto g \not\circ E$ are opening operators on $\mathcal{Q}(X)$.

Since F is reflexive and transitive, we have $g \circ F \circ F = g \circ F$ for each $g \in \mathcal{Q}(X)$. Consequently, $g \circ F \in \langle w \circ F = w \rangle = \langle w \circ R_i \subseteq w \mid i \in I \rangle$. Moreover, if $h \in \langle w \circ R_i \subseteq w \mid i \in I \rangle$ such that $g \subseteq h$, then $g \circ F \subseteq h \circ F = h$. Hence, $g \circ F$ represents the least solution to the system $w \circ R_i \subseteq w$, for $i \in I$, that contains g . That means $g \circ F = g^{lc}$.

The remaining equalities in (77) can be proved analogously. $\square$

The Proposition 4.1 provides effective methods for computing the greatest solutions to the systems (77) that are contained in a given hesitant fuzzy set g on X , as well as the least solutions that contain g . A method for determining the greatest solution to the system $w \circ R_i \subseteq w$ ($i \in I$) contained in a given hesitant fuzzy set g is presented in Algorithm 1. Analogous algorithms can be formulated for the other systems of inequalities that we have considered.

Algorithm 1 Computing the greatest solution to the system of inequalities $w \circ R_i \subseteq w$ ($i \in I$) contained in g

Input: A family $\{R_i\}_{i \in I}$ of hesitant fuzzy relations on a finite set X (with finite index set I) and a hesitant fuzzy set g on X .

Output: The greatest solution to the system of inequalities $w \circ R_i \subseteq w$ ($i \in I$) contained in g .

- 1: Compute: $R = \bigcup_{i \in I} R_i$
 - 2: Compute the reflexive-transitive closure: $F \leftarrow R^f$
 - 3: Compute the left $\circ$ -residual: $L \leftarrow g \not\circ F$
 - 4: **return** L
-

5 Hesitant fuzzy relation equations

In this section, we investigate hesitant fuzzy relation equations and present a detailed analysis of their solution sets.

Theorem 5.1. Let $\{R_i\}_{i \in I}$ be a family of hesitant fuzzy relations on X and let $R = \bigcup_{i \in I} R_i$. Then the following holds:

$$\langle w \circ R_i = w \mid i \in I \rangle \subseteq \langle w \circ R = w \rangle, \quad \langle R_i \circ w = w \mid i \in I \rangle \subseteq \langle R \circ w = w \rangle. \quad (78)$$

Proof. Assume that $g \in \langle w \circ R_i = w \mid i \in I \rangle$, i.e., $g \circ R_i = g$ for all $i \in I$. Then, by applying property (54), we obtain:

$$g \circ R = g \circ \left(\bigcup_{i \in I} R_i \right) = \bigcup_{i \in I} (g \circ R_i) = g.$$

Therefore, $g \in \langle w \circ R = w \rangle$ and thus the first inclusion in (78) is proved. The second inclusion can be proved similarly. $\square$

The following example illustrates that the inclusions presented in (78) can be strict.

Example 5.1. Let $X = \{x, y, z\}$, and let $\mathcal{Q}$ be a Gödel power set quantale. Consider the hesitant fuzzy relations $R_1, R_2 \in \mathcal{R}(X)$ and the hesitant fuzzy set $g \in \mathcal{Q}(X)$, given by:

$$R_1 = \begin{bmatrix} \{0.1, 0.2\} & [0, 1] & \{0.3\} \\ \{0.4\} & \emptyset & \{0.5, 0.6\} \\ \{0.1, 0.2\} & [0, 1] & \emptyset \end{bmatrix}, \quad R_2 = \begin{bmatrix} \{0.1\} & \{0.5\} & \{0.2\} \\ \{0.2\} & \emptyset & [0, 1] \\ \{0.3\} & \{0.4\} & \emptyset \end{bmatrix}, \quad g = \begin{bmatrix} [0, 1] \\ [0, 1] \\ [0, 1] \end{bmatrix}.$$

In that case, we have:

$$R = R_1 \cup R_2 = \begin{bmatrix} \{0.1, 0.2\} & [0, 1] & \{0.2, 0.3\} \\ \{0.2, 0.4\} & \emptyset & [0, 1] \\ \{0.1, 0.2, 0.3\} & [0, 1] & \emptyset \end{bmatrix}.$$

Consequently, $g \circ R = g$ and $g \in \langle w \circ R = w \rangle$. However,

$$g \circ R_2 = \begin{bmatrix} [0, 0.3] \\ [0, 0.5] \\ [0, 1] \end{bmatrix} \neq g,$$

whence it follows that $g \notin \langle w \circ R_i = w \mid i \in I \rangle$.

Theorem 5.2. Let R be a hesitant fuzzy relation on X . The following holds:

$$\langle w \circ R = w \rangle = \langle w \circ R^t = w \rangle, \quad \langle R \circ w = w \rangle = \langle R^t \circ w = w \rangle. \quad (79)$$

Proof. Assume that $g \in \langle w \circ R = w \rangle$, i.e., $g \circ R = g$. We show by mathematical induction that $g \circ R^n = g$ for all $n \in \mathbb{N}$. The base case $n = 1$ holds by the assumption $g \circ R = g$. If $g \circ R^n = g$, for some $n \in \mathbb{N}$, then the associativity of composition implies:

$$g \circ R^{n+1} = (g \circ R^n) \circ R = g \circ R = g.$$

Hence, by induction, the equality $g \circ R^n = g$ holds for all $n \in \mathbb{N}$. Consequently, we have:

$$g \circ R^t = g \circ \left(\bigcup_{n \in \mathbb{N}} R^n \right) = \bigcup_{n \in \mathbb{N}} g \circ R^n = g,$$

that is, $g \in \langle w \circ R^t = w \rangle$. It follows that $\langle w \circ R = w \rangle \subseteq \langle w \circ R^t = w \rangle$.

Conversely, let $g \in \langle w \circ R^t = w \rangle$. Then $g \circ R \subseteq g \circ R^t = g$, and we obtain:

$$g \circ R^2 = (g \circ R) \circ R \subseteq g \circ R.$$

Assume that $g \circ R^n \subseteq g \circ R^{n-1}$ for some $n \in \mathbb{N}$. Then:

$$g \circ R^{n+1} = (g \circ R^n) \circ R \subseteq g \circ R^{n-1} \circ R = g \circ R^n.$$

By induction, it follows that $g \circ R^n \subseteq g \circ R^{n-1}$ for all $n \in \mathbb{N}$, so $\{(g \circ R^n)\}_{n \in \mathbb{N}}$ is a decreasing sequence. Hence, its join is the first element, giving:

$$g = g \circ R^t = g \circ \left(\bigcup_{n \in \mathbb{N}} R^n \right) = \bigcup_{n \in \mathbb{N}} (g \circ R^n) = g \circ R.$$

Consequently, $g \in \langle w \circ R = w \rangle$, which implies the inclusion $\langle w \circ R^t = w \rangle \subseteq \langle w \circ R = w \rangle$. Together with the previously proved reverse inclusion, it follows that $\langle w \circ R = w \rangle = \langle w \circ R^t = w \rangle$. The other equality in (79) can be proved analogously. $\square$

Theorem 5.3. *Let $\{R_i\}_{i \in I}$ be a family of hesitant fuzzy relations on X . Then*

$$\langle w \circ R_i = w \mid i \in I \rangle, \quad \langle R_i \circ w = w \mid i \in I \rangle, \quad \langle w \circ R_i = w, R_i \circ w = w \mid i \in I \rangle \quad (80)$$

are opening systems in $\mathcal{Q}(X)$.

Proof. To simplify notation we set $\mathcal{C} = \langle w \circ R_i = w \mid i \in I \rangle$. It follows directly that $\emptyset \in \mathcal{C}$. Next, let $\{g_j\}_{j \in J}$ be an arbitrary family of hesitant fuzzy sets on X such that $\{g_j\}_{j \in J} \subseteq \mathcal{C}$. Then, for every $i \in I$ we have $g_j \circ R_i = g_j$ for all $j \in J$. Consequently, by property (56):

$$\left(\bigcup_{j \in J} g_j \right) \circ R_i = \bigcup_{j \in J} (g_j \circ R_i) = \bigcup_{j \in J} g_j, \quad (81)$$

which implies that $\bigcup_{j \in J} g_j \in \mathcal{C}$. Hence, $\mathcal{C} = \langle w \circ R_i = w \mid i \in I \rangle$ is an opening system in $\mathcal{Q}(X)$.

The proof for the remaining systems in (80) is analogous and therefore omitted. $\square$

The following example demonstrates that the systems given in (80) do not always form closure systems.

Example 5.2. Let $X = \{x, y, z\}$, and let $\mathcal{Q}$ be the Gödel power set quantale. Consider the hesitant fuzzy relations R_1 and R_2 on X , given in Example 5.1, and the full hesitant fuzzy set $h_S = [[0, 1][0, 1][0, 1]]^T$ on X . Then, we obtain:

$$h_S \circ R_2 = \begin{bmatrix} [0, 0.2] \\ [0, 1] \\ [0, 0.6] \end{bmatrix} \neq h_S.$$

It follows that h_S is not a solution to the systems $\langle w \circ R_2 = w \rangle$ and $\langle w \circ R_1 = w, w \circ R_2 = w \rangle$, defined in (80). Consequently, these systems do not constitute closure systems.

In addition to the hesitant fuzzy relation inequalities specified in (71), the term hesitant fuzzy relation inequality can more generally refer to any expression of the form:

$$w \subseteq \varphi(w), \quad (82)$$

where w is unknown hesitant fuzzy set taking values in $\mathcal{Q}^X$ and $\varphi : \mathcal{Q}^X \mapsto \mathcal{Q}^X$ is a function. Any hesitant fuzzy set $g \in \mathcal{Q}^X$ satisfying the inequality $g \subseteq \varphi(g)$ is considered as a solution to that inequality. In order to establish a connection between the solution sets of the inequalities given in (71) and those corresponding to the more general expression in (82), we define the following functions:

$$\begin{aligned} \varphi^l(g) &= \bigcap_{i \in I} (g \circ R_i \cap g \not\subseteq R_i), \\ \varphi^r(g) &= \bigcap_{i \in I} (R_i \circ g \cap R_i \not\subseteq g), \\ \varphi^d(g) &= \bigcap_{i \in I} (g \circ R_i \cap g \not\subseteq R_i \cap R_i \circ g \cap R_i \not\subseteq g) = (\varphi^l \cap \varphi^r)(g). \end{aligned} \quad (83)$$

The following theorem shows that the functions φ^l , φ^r and φ^d are isotone, and it characterizes the relationship between the sets of solutions to the inequalities defined in (71) and those given in (82).

Theorem 5.4. *Let $\{R_i\}_{i \in I}$ be a family of hesitant fuzzy relations on X . Let $\varphi^l, \varphi^r, \varphi^d$ be functions on $\mathcal{Q}(X)$, defined for any $g \in \mathcal{Q}^X$ as in (83). Then, the functions $\varphi^l, \varphi^r, \varphi^d$ are isotone and the following holds:*

$$\begin{aligned} \langle w \circ R_i = w \mid i \in I \rangle &= \langle w \subseteq \varphi^l(w) \rangle, & \langle R_i \circ w = w \mid i \in I \rangle &= \langle w \subseteq \varphi^r(w) \rangle, \\ \langle w \circ R_i = w, R_i \circ w = w \mid i \in I \rangle &= \langle w \subseteq \varphi^d(w) \rangle. \end{aligned} \quad (84)$$

Proof. The isotonicity of the functions φ^l, φ^r and φ^d follows directly from Proposition 3.2.

Consider the first equality in (84). Based on the previous results, we establish the following sequence of equalities:

$$\begin{aligned} \langle w \circ R_i = w \mid i \in I \rangle &= \langle w \circ R_i \subseteq w, w \subseteq w \circ R_i \mid i \in I \rangle = \langle w \subseteq w \not\subseteq R_i, w \subseteq w \circ R_i \mid i \in I \rangle \\ &= \langle w \subseteq (w \not\subseteq R_i \cap w \circ R_i) \mid i \in I \rangle = \left\langle w \subseteq \bigcap_{i \in I} (w \not\subseteq R_i \cap w \circ R_i) \right\rangle = \langle w \subseteq \varphi^l(w) \rangle. \end{aligned}$$

This completes the proof of the first equality in (84), and the remaining equalities follow by analogous arguments. $\square$

We proved that functions φ^l, φ^r and φ^d are isotone and it can be easily shown that the sets of all solutions to the inequalities $g \subseteq \varphi^l(g)$, $g \subseteq \varphi^r(g)$ and $g \subseteq \varphi^d(g)$ are complete sublattices of $\mathcal{Q}^X$, containing the greatest post fixed points of the corresponding functions. Considering that all the conditions of Knaster-Tarski fixed point theorem are fulfilled, we have that the greatest fixed points of $\varphi^l, \varphi^r, \varphi^d$ are equal to the greatest post fix points of these functions, respectively. This allows us to compute the greatest solutions to the given inequalities (equations) contained in an arbitrary hesitant fuzzy set g of $\mathcal{Q}^X$.

Recall that the image $Im(g)$ of a hesitant fuzzy set g on X is the set of all membership degrees that appear in g , i.e., $Im(g) = \{g(x) \mid x \in X\}$. Similarly, the image of a hesitant fuzzy relation R_i is the set of all membership degrees assigned by R_i to the ordered pairs in $X \times X$, i.e., $Im(R_i) = \{R_i(x, y) \mid x, y \in X\}$. The following theorem can be stated.

Theorem 5.5. *Assume that X and I are finite sets and let $\{R_i\}_{i \in I}$ be a collection of hesitant fuzzy relations on X . Let g be a hesitant fuzzy set on X . Construct a sequence $\{g_k\}_{k \in \mathbb{N}}$ of hesitant fuzzy sets on X inductively as follows:*

$$g_1 = g, \quad g_{k+1} = g_k \cap \varphi^l(g_k), \quad \text{for each } k \in \mathbb{N}.$$

If the subalgebra of $\mathcal{Q}$, generated by the set $\bigcup_{i \in I} (Im(R_i)) \cup Im(g)$ is finite, then the following assertions hold:

1. *The sequence $\{g_k\}_{k \in \mathbb{N}}$ is finite and decreasing, and there is the smallest number $m \in \mathbb{N}$ such that $g_m = g_{m+1}$,*
2. *g_m is the greatest solution to the system $w \circ R_i = w$ ($i \in I$) contained in g .*

Proof. Let $\mathcal{D}$ denote the subalgebra $\bigcup_{i \in I} (Im(R_i)) \cup Im(g)$ and let D denote the carrier set of $\mathcal{D}$. By the hypothesis, D is a finite set, so the set D^X of all hesitant fuzzy sets on X with membership values in $\mathcal{D}$ is also finite. Since $\{g_k\}_{k \in \mathbb{N}} \subseteq D^X$, the sequence $\{g_k\}_{k \in \mathbb{N}}$ is also finite. Moreover, $\{g_k\}_{k \in \mathbb{N}}$ clearly forms a decreasing sequence, and for each $k \in \mathbb{N}$, we have $g_k \subseteq g$.

A finite decreasing sequence must stabilize; that is, starting from some index, all elements are equal. Consequently, there exist $m, n \in \mathbb{N}$ such that $g_{m+n} = g_m$, and then $g_m = g_{m+n} \subseteq g_{m+1} \subseteq g_m$. Hence, $g_{m+1} = g_m$, and m is the smallest natural number with this property. Therefore, we obtain:

$$g_m = g_{m+1} = g_m \cap \varphi^l(g_m),$$

and hence $g_m \subseteq \varphi^l(g_m)$. By Theorem 5.4, g_m is a solution to the system $w \circ R_i = w$ ($i \in I$).

Consider an arbitrary solution h to this system, satisfying $h \subseteq g$. Then we have $h \subseteq \varphi^l(h)$ and $h \subseteq g = g_1$. If $h \subseteq g_k$, for some $k \in \mathbb{N}$, then $\varphi^l(h) \subseteq \varphi^l(g_k)$. Hence, the following holds:

$$h \subseteq g_k \cap \varphi^l(h) \subseteq g_k \cap \varphi^l(g_k) = g_{k+1}.$$

Therefore, since $h \subseteq g_k$ for each $k \in \mathbb{N}$, then $h \subseteq g_m$. Consequently, g_m is the greatest solution to the system $w \circ R_i = w$ ($i \in I$) contained in g . $\square$

Remark 5.1. The Kleene fixed point theorem asserts that the greatest fixed point of the given function φ^l can be computed as the intersection of the descending Kleene chain $\{g_k\}_{k \in \mathbb{N}}$, which is defined in Theorem 5.5. Observe that the previous theorem gives sufficient conditions for a given descending sequence to be stabilized and the greatest solution to the system $w \circ R_i = w$ ($i \in I$) contained in g can be computed in this way. Therefore, the result can alternatively be interpreted within this fixed point framework.

Algorithm 2 Computing the greatest solution to the system of equations $w \circ R_i = w$ ($i \in I$) contained in g

Input: A family $\{R_i\}_{i \in I}$ of hesitant fuzzy relations on a finite set X (with finite index set I) and a hesitant fuzzy set g on X .

Output: The greatest solution to the system of equations $w \circ R_i = w$ ($i \in I$) contained in g .

```

1: function COMPUTE  $\varphi^l(g, \{R_i\}_{i \in I})$ 
2:   Initialize all values of HFS  $f$  to  $[0, 1]$ 
3:   for each  $i \in I$  do
4:      $h \leftarrow (g \circ R_i) \cap (g \not\circ R_i)$ 
5:      $f \leftarrow f \cap h$ 
6:   end for
7:   return  $f$ 
8: end function
9:  $g_{\text{current}} \leftarrow g$ 
10: do
11:    $g_{\text{previous}} \leftarrow g_{\text{current}}$ 
12:    $g_{\text{current}} \leftarrow g_{\text{previous}} \cap \text{COMPUTE } \varphi^l(g_{\text{previous}}, \{R_i\}_{i \in I})$ 
13: while  $g_{\text{current}} \neq g_{\text{previous}}$ 
14: return  $g_{\text{current}}$ 
```

Consider a system represented by the single equation $w \circ R = w$. In order to compute the greatest solution to this equation contained in a hesitant fuzzy set g on X , we first determine the greatest solution g' contained in g to the inequality $w \circ R \subseteq w$ and then compute the greatest solution to the inequality $w \subseteq w \circ R$ contained in g' (i.e., the greatest post-fixed point of the function $h \mapsto h \circ R$ contained in g'). The obtained hesitant fuzzy set is also the greatest solution to the equation $w \circ R = w$ contained in g .

Theorem 5.6. Let R be a hesitant fuzzy relation on a finite universal set X , let $F = R^f$, and let g be a hesitant fuzzy set on X . Construct a sequence $\{h_k\}_{k \in \mathbb{N}}$ of hesitant fuzzy sets on X inductively as follows:

$$h_1 = g \not\circ F, \quad h_{k+1} = h_k \circ R, \quad \text{for each } k \in \mathbb{N}. \quad (85)$$

If the subsemiring of $\mathcal{Q}^*$ generated by the set $\text{Im}(R) \cup \text{Im}(g)$ is finite, then the following holds:

1. The sequence $\{h_k\}_{k \in \mathbb{N}}$ is finite and decreasing, and there is the smallest number $m \in \mathbb{N}$ such that $h_m = h_{m+1}$,
2. h_m is the greatest solution to equation $w \circ R = w$ contained in g .

Proof. According to Proposition 4.1, $h_1 = g \not\circ F$ is the greatest solution to the inequality $w \circ R \subseteq w$ contained in g . Hence, $h_1 \circ R \subseteq h_1$ and, therefore, $h_2 = h_1 \circ R \subseteq h_1$. Next, assume that $h_{k+1} \subseteq h_k$ for some $k \in \mathbb{N}$. Then:

$$h_{k+2} = h_{k+1} \circ R \subseteq h_k \circ R = h_{k+1}.$$

By induction, $h_{k+1} \subseteq h_k$ holds for any $k \in \mathbb{N}$. Therefore, $\{h_k\}_{k \in \mathbb{N}}$ is a decreasing sequence, and $h_k \subseteq g \not\circ F \subseteq g$, for any $k \in \mathbb{N}$. Similarly to the proof of Theorem 5.5, $\{h_k\}_{k \in \mathbb{N}}$ is a finite sequence, so there exists the smallest $m \in \mathbb{N}$ such that $h_m = h_{m+1}$.

Since $h_m = h_{m+1} = h_m \circ R$, then h_m is a solution to the equation $w \circ R = w$. Assume h is an arbitrary solution to this equation contained in g . Then h is also a solution to the inequality $w \circ R \subseteq w$. According to Proposition 4.1, $g \not\circ F$ is the greatest solution to the inequality $w \circ R \subseteq w$. Therefore, we have $h \subseteq g \not\circ F$ and $(g \not\circ F) \circ R \subseteq (g \not\circ F)$. Consequently,

$$h = h \circ R \subseteq (g \not\circ F) \circ R \subseteq g \not\circ F = h_1.$$

If $h \subseteq h_k$, for some $k \in \mathbb{N}$, then

$$h = h \circ R \subseteq h_k \circ R = h_{k+1}.$$

Therefore, by induction, we obtain $h \subseteq h_k$, for every $k \in \mathbb{N}$, and consequently, $h \subseteq h_m$. Hence, h_m is the greatest solution to the equation $w \circ R = w$ contained in g . $\square$

Remark 5.2. The iterative construction in Theorem 5.6 can also be viewed as the iteration of an isotone operator on a complete lattice of hesitant fuzzy sets, and therefore Knaster-Tarski fixed point theorem can be applied. The sequence $\{h_k\}_{k \in \mathbb{N}}$ is decending (Kleene's chain) and if the conditions of the previous theorem are fulfilled, then the sequence stabilizes, i.e. there exists $m \in \mathbb{N}$ such that $h_m = h_{m+1}$. Hence, h_m is the greatest fixed point of the given operator $\varphi(h_k) = h_k \circ R$, i.e., the greatest solution to the equation $w \circ R = w$ contained in g .

Algorithm 3 Computing the greatest solution to the equation $w \circ R = w$ contained in g

Input: A hesitant fuzzy relation R on a finite set X and a hesitant fuzzy set g on X .

Output: The greatest solution to the equation $w \circ R = w$ contained in g .

- 1: Compute the reflexive-transitive closure: $F \leftarrow R^f$
 - 2: Compute the left residual: $L \leftarrow g \not\circ F$
 - 3: Set $h_{\text{current}} \leftarrow L$
 - 4: **do**
 - 5: $h_{\text{previous}} \leftarrow h_{\text{current}}$
 - 6: $h_{\text{current}} \leftarrow h_{\text{previous}} \circ R$
 - 7: **while** $h_{\text{current}} \neq h_{\text{previous}}$
 - 8: **return** h_{current}
-

Note that the previous algorithm provides an alternative approach when the system consists of a single equation. The algorithm requires only two variables, the input set g and a second variable to store each newly computed value. However, for the sake of readability and more intuitive notation, we have introduced an additional variable.

6 Algebraic characterizations of the sets of solutions

This section is devoted to analyzing families of hesitant fuzzy sets that form the solution sets to hesitant fuzzy relation equations, specifically when the relation is a hesitant fuzzy quasi-order. We first establish several properties of closure systems on families of hesitant fuzzy sets, which provide the foundation for the subsequent analysis.

Let $\mathcal{C}$ be a closure system in $\mathcal{Q}(X)$. We say that the principal elements C_x and C_y of $\mathcal{C}$ satisfy:

- the *inclusion property* if

$$C_x(y) = I(C_x, C_y) = \bigcap_{a \in X} C_x(a) \rightarrow C_y(a), \quad (86)$$

- the *reverse inclusion property* if

$$C_x(y) = I(C_y, C_x) = \bigcap_{a \in X} C_y(a) \rightarrow C_x(a), \quad (87)$$

- the *equality property* if

$$C_x(y) = E(C_y, C_x) = \bigcap_{a \in X} C_x(a) \leftrightarrow C_y(a), \quad (88)$$

for all $x, y \in X$.

In what follows, we present a theorem proved in [24], which will be used in proving the subsequent results. Recall that for a hesitant fuzzy quasi-order F on X , the hesitant fuzzy equivalence E_F on X , defined by $E_F(x, y) = F(x, y) \cap F(y, x)$, is referred to as the natural hesitant fuzzy equivalence of F .

Theorem 6.1. *Let F be a hesitant fuzzy quasi-order on a set X . In that case*

1. E_F is the greatest hesitant fuzzy equivalence on X contained in F .
2. For any $x, y \in X$

$$F(x, y) = \bigcap_{a \in X} aF(x) \rightarrow aF(y) = \bigcap_{a \in X} Fa(y) \rightarrow Fa(x). \quad (89)$$

3. For any $x, y \in X$

$$E_F(x, y) = \bigcap_{a \in X} aF(x) \leftrightarrow aF(y) = \bigcap_{a \in X} Fa(x) \leftrightarrow Fa(y). \quad (90)$$

Proposition 6.1. A family $\mathcal{P}$ of normalized hesitant fuzzy sets on X is the collection of all aftersets of some hesitant fuzzy quasi-order on X if and only if $\mathcal{P}$ is the principal part of some closure system in $\mathcal{Q}(X)$ whose principal elements satisfy the reverse inclusion property.

Proof. Let $\mathcal{P}$ be the collection of all aftersets of F . Define a collection $\mathcal{C} \subseteq \mathcal{Q}(X)$ in the following way:

$$\mathcal{C} = \langle w \circ F \subseteq w \rangle = \{g \in \mathcal{Q}(X) \mid F \subseteq g \setminus g\}.$$

Recall that for any $x \in X$, the afterset xF is defined by $xF(y) = F(x, y)$ for all $y \in X$. Consequently,

$$xF(y) \circ F(x, y) \subseteq xF(y)$$

for all $x, y \in X$, which implies that $xF \in \mathcal{C}$ for all $x \in X$. By Theorem 4.5, $\mathcal{C}$ is a closure system in $\mathcal{Q}(X)$.

Now, fix an arbitrary $a \in X$ and define the set $K = \{g \in \mathcal{C} \mid 1_\otimes \subseteq g(a)\}$. By Lemma 3.2, we obtain:

$$\mathcal{C}_a(x) = \bigcap_{g \in K} g(x) \supseteq \bigcap_{g \in K} (g(a) \rightarrow g(x)) \supseteq \bigcap_{g \in \mathcal{C}} (g(a) \rightarrow g(x)) = \bigcap_{g \in \mathcal{C}} (g \setminus g)(a, x) \supseteq F(a, x) = aF(x),$$

for any $x \in X$. Hence, $aF \subseteq \mathcal{C}_a$. On the other hand, if $aF \in K$ it follows that:

$$\mathcal{C}_a = \bigcap_{g \in K} g \subseteq aF.$$

Therefore, $\mathcal{C}_a = aF$ for all $a \in X$, which implies that $\mathcal{P}$ represents the principal part of $\mathcal{C}$. According to Theorem 6.1, the following holds:

$$\bigcap_{x \in X} \mathcal{C}_b(x) \rightarrow \mathcal{C}_a(x) = \bigcap_{x \in X} bF(x) \rightarrow aF(x) = \bigcap_{x \in X} Fx(b) \rightarrow Fx(a) = F(a, b) = aF(b) = \mathcal{C}_a(b),$$

for all $a, b \in X$. Hence, principal elements of $\mathcal{C}$ satisfy the reverse inclusion property.

For the opposite direction, assume that $\mathcal{P}$ is the principal part of some closure system $\mathcal{C}$ in $\mathcal{Q}(X)$ whose principal elements satisfy the reverse inclusion property. Define a hesitant fuzzy relation F on X in the following way:

$$F(a, b) = \mathcal{C}_a(b) = I(\mathcal{C}_b, \mathcal{C}_a) = \bigcap_{x \in X} \mathcal{C}_b(x) \rightarrow \mathcal{C}_a(x),$$

for all $a, b \in X$. By definition of the principal element $\mathcal{C}_x$ of the closure system $\mathcal{C}$ generated by $x \in X$ (cf. (45)), we have $1_\otimes \subseteq \mathcal{C}_x(x)$ for every $x \in X$. Consequently, as $F(a, a) = \mathcal{C}_a(a)$, the relation F is reflexive.

Consider arbitrary $a, b, c \in X$. Using property (12), we obtain:

$$F(a, b) \otimes F(b, c) \subseteq (\mathcal{C}_b(x) \rightarrow \mathcal{C}_a(x)) \otimes (\mathcal{C}_c(x) \rightarrow \mathcal{C}_b(x)) \subseteq \mathcal{C}_c(x) \rightarrow \mathcal{C}_a(x), \quad (91)$$

for each $x \in X$. Consequently:

$$F(a, b) \otimes F(b, c) \subseteq \bigcap_{x \in X} \mathcal{C}_c(x) \rightarrow \mathcal{C}_a(x) = F(a, c).$$

Hence, F is a hesitant fuzzy quasi-order. According to (91), it holds $(aF)(b) = F(a, b) = \mathcal{C}_a(b)$ for all $a, b \in X$, and it follows that $aF = \mathcal{C}_a$ for all $a \in X$. Therefore, $\mathcal{P}$ consists of all F -aftersets. $\square$

The corresponding properties of the aforementioned family $\mathcal{P}$, expressed in terms of foresets (or equivalence classes) and inclusion (or equality), are presented below and can be proved similarly to Proposition 6.1.

Proposition 6.2. A family $\mathcal{P}$ of normalized hesitant fuzzy sets on X is the collection of all foresets of some hesitant fuzzy quasi-order on X if and only if $\mathcal{P}$ is the principal part of some closure system in $\mathcal{Q}(X)$ whose principal elements satisfy the inclusion property.

Proposition 6.3. A family $\mathcal{P}$ of normalized hesitant fuzzy sets on X is the collection of all equivalence classes of some hesitant fuzzy quasi-order on X if and only if $\mathcal{P}$ is the principal part of some closure system in $\mathcal{Q}(X)$ whose principal elements satisfy the equality property.

As previously mentioned, the lattice of all hesitant fuzzy sets on X , equipped with the scalar multiplication is denoted by $\mathcal{Q}_{\otimes}(X)$, called $\mathcal{Q}$ -lattice of hesitant fuzzy sets on X . Its complete sublattices equipped also with the scalar multiplication are said to be complete $\mathcal{Q}$ -sublattices of $\mathcal{Q}_{\otimes}(X)$.

Theorem 6.2. Let $\mathcal{C}$ be a collection of hesitant fuzzy sets on X . Then there exists a hesitant fuzzy quasi-order F on X such that $\mathcal{C} = \langle w \circ F = w \rangle$ if and only if the following hold:

1. $\mathcal{C}$ is a complete $\mathcal{Q}$ -sublattice of $\mathcal{Q}_{\otimes}(X)$;
2. every element of $\mathcal{C}$ can be expressed as a linear combination of principal elements of $\mathcal{C}$;
3. principal elements of $\mathcal{C}$ satisfy the reverse inclusion property.

Proof. Let F be a hesitant fuzzy quasi-order on X and let $\mathcal{C} = \langle w \circ F = w \rangle$. By properties (56), (57), and using the reflexivity of F , for any family $\{g_i\}_{i \in I} \subseteq \mathcal{C}$ we have:

$$\left(\bigcup_{i \in I} g_i\right) \circ F = \bigcup_{i \in I} (g_i \circ F) = \bigcup_{i \in I} g_i, \quad \left(\bigcap_{i \in I} g_i\right) \circ F \subseteq \bigcap_{i \in I} (g_i \circ F) = \bigcap_{i \in I} g_i \subseteq \left(\bigcap_{i \in I} g_i\right) \circ F.$$

Consequently, $\mathcal{C}$ is closed under arbitrary joins and meets. Let now $\lambda \in Q$ be an arbitrary scalar and $g \in \mathcal{C}$. For any $x \in X$, using the properties of scalar multiplication, we obtain:

$$((\lambda \otimes g) \circ F)(x) = \bigcup_{y \in X} ((\lambda \otimes g)(y) \otimes F(y, x)) = \lambda \otimes \bigcup_{y \in X} (g(y) \otimes F(y, x)) = \lambda \otimes (g \circ F)(x) = \lambda \otimes g(x).$$

Thus, $\mathcal{C}$ is closed under scalar multiplication. Consequently, $\mathcal{C}$ forms a complete $\mathcal{Q}$ -sublattice of $\mathcal{Q}_{\otimes}(X)$.

Next, fix an arbitrary $x \in X$. Since F is a hesitant fuzzy-quasi order, the following holds:

$$(xF \circ F)(y) = \bigcup_{z \in X} (xF(z) \otimes F(z, y)) = \bigcup_{z \in X} (F(x, z) \otimes F(z, y)) = (F \circ F)(x, y) = xF(y),$$

for all $y \in X$. Hence, $xF \in \mathcal{C}$ for all $x \in X$. Since F is reflexive, it follows that $1_{\otimes} \subseteq xF(x)$ for all $x \in X$. Therefore, $xF \in \{g \in \mathcal{C} \mid 1_{\otimes} \subseteq g(x)\}$, which implies that $\{g \in \mathcal{C} \mid 1_{\otimes} \subseteq g(x)\}$ is non-empty set. Consider now an arbitrary $g \in \mathcal{C}$ such that $1_{\otimes} \subseteq g(x)$ for some $x \in X$. Then for any $y \in X$ we have:

$$g(y) = (g \circ F)(y) = \bigcup_{z \in X} (g(z) \otimes F(z, y)) \supseteq g(x) \otimes F(x, y) \supseteq F(x, y) = xF(y),$$

and it follows that $xF \subseteq g$. Consequently, $xF \subseteq g$ for all $g \in \mathcal{C}$, and, hence, xF is the least element of the set $\{g \in \mathcal{C} \mid 1_{\otimes} \subseteq g(x)\}$. Therefore, $xF = \bigcap \{g \in \mathcal{C} \mid 1_{\otimes} \subseteq g(x)\} = \mathcal{C}_x$ for all $x \in X$. According to (89), for arbitrary $x, y \in X$ we obtain:

$$\bigcap_{a \in X} \mathcal{C}_y(a) \rightarrow \mathcal{C}_x(a) = \bigcap_{a \in X} yF(a) \rightarrow xF(a) = \bigcap_{a \in X} Fa(y) \rightarrow Fa(x) = F(x, y) = \mathcal{C}_x(y).$$

Therefore, principal elements of $\mathcal{C}$ satisfy the reverse inclusion property.

Consider now an arbitrary $g \in \mathcal{C}$. The following holds:

$$g(x) = (g \circ F)(x) = \bigcup_{y \in X} (g(y) \otimes F(y, x)) = \bigcup_{y \in X} (g(y) \otimes \mathcal{C}_y(x)) = \left(\bigcup_{y \in X} g(y) \otimes \mathcal{C}_y\right)(x)$$

for all $x \in X$. Hence:

$$g = \bigcup_{y \in X} g(y) \otimes \mathcal{C}_y,$$

and, consequently, every member of $\mathcal{C}$ can be represented as a linear combination of its principal elements. Therefore, we proved that each of the properties 1., 2., and 3. holds.

Conversely, assume that $\mathcal{C}$ satisfies conditions 1., 2., and 3. Define a hesitant fuzzy relation F on X in the following way:

$$F(x, y) = \mathcal{C}_x(y) = \bigcap_{a \in X} \mathcal{C}_y(a) \rightarrow \mathcal{C}_x(a), \quad (92)$$

for all $x, y \in X$. It is easy to see that F is reflexive. Consider arbitrary $x, y, z \in X$. According to (92) the following holds:

$$\begin{aligned} F(x, y) \otimes F(y, z) &= \left(\bigcap_{a \in X} \mathcal{C}_y(a) \rightarrow \mathcal{C}_x(a) \right) \otimes \left(\bigcap_{a \in X} \mathcal{C}_z(a) \rightarrow \mathcal{C}_y(a) \right) \\ &\subseteq (\mathcal{C}_y(a) \rightarrow \mathcal{C}_x(a)) \otimes (\mathcal{C}_z(a) \rightarrow \mathcal{C}_y(a)). \end{aligned}$$

Next, by property (12) we have:

$$(\mathcal{C}_y(a) \rightarrow \mathcal{C}_x(a)) \otimes (\mathcal{C}_z(a) \rightarrow \mathcal{C}_y(a)) \subseteq \mathcal{C}_z(a) \rightarrow \mathcal{C}_x(a).$$

Hence, for all $a \in X$ it holds $F(x, y) \otimes F(y, z) \subseteq \mathcal{C}_z(a) \rightarrow \mathcal{C}_x(a)$ and it follows:

$$F(x, y) \otimes F(y, z) \subseteq \bigcap_{a \in X} \mathcal{C}_z(a) \rightarrow \mathcal{C}_x(a) = F(x, z).$$

Therefore, F is transitive, and, consequently, F is a hesitant fuzzy quasi-order. By the definition of F , for an arbitrary $x \in X$, we have $\mathcal{C}_x = xF$.

Consider now an arbitrary $g \in \langle w \circ F = w \rangle$. Then for each $x \in X$ the following holds:

$$g(x) = \bigcup_{y \in X} (g(y) \otimes F(y, x)) = \bigcup_{y \in X} (g(y) \otimes \mathcal{C}_y(x)) = \left(\bigcup_{y \in X} g(y) \otimes \mathcal{C}_y \right)(x).$$

Consequently, we obtain:

$$g = \bigcup_{y \in X} g(y) \otimes \mathcal{C}_y.$$

Since $\mathcal{C}_y \in \mathcal{C}$ for all $y \in X$, and $\mathcal{C}$ is closed under scalar products and arbitrary joins, then it follows that $g \in \mathcal{C}$. Therefore, $\langle w \circ F = w \rangle \subseteq \mathcal{C}$.

On the other hand, let $g \in \mathcal{C}$. According to assertion 2., there are scalars $\lambda_z \in Q$, $z \in X$, such that

$$g = \bigcup_{z \in X} \lambda_z \otimes \mathcal{C}_z,$$

and, for any $x \in X$, the following holds:

$$\begin{aligned} (g \circ F)(x) &= \bigcup_{y \in X} (g(y) \otimes F(y, x)) = \bigcup_{y \in X} \left(\bigcup_{z \in X} \lambda_z \otimes \mathcal{C}_z(y) \right) \otimes F(y, x) \\ &= \bigcup_{z \in X} \left(\lambda_z \otimes \left(\bigcup_{y \in X} F(z, y) \otimes F(y, x) \right) \right) = \bigcup_{z \in X} \left(\lambda_z \otimes (F \circ F(z, x)) \right) \\ &= \bigcup_{z \in X} (\lambda_z \otimes \mathcal{C}_z(x)) = g(x). \end{aligned}$$

Thus, $g \in \langle w \circ F = w \rangle$, so $\mathcal{C} \subseteq \langle w \circ F = w \rangle$. Hence, it has been proved that $\mathcal{C} = \langle w \circ F = w \rangle$. $\square$

Recall that the set $\mathcal{F}(X)$ of all hesitant fuzzy quasi-orders on X is a closure system in the lattice $\mathcal{R}(X)$ of all hesitant fuzzy relations on X . Thus, $\mathcal{F}(X)$ is a complete lattice under inclusion.

The above results enables the construction of an order isomorphism between the lattice $\langle \mathcal{F}(X), \subseteq \rangle$ and a partially ordered set of particular complete $\mathcal{Q}$ -sublattice of $\mathcal{Q}_{\otimes}(X)$, implying that the latter inherits the lattice structure. We explore this relationship below.

Theorem 6.3. *The mapping $F \mapsto \langle w \circ F = w \rangle$ is a dual order isomorphism of the lattice $\mathcal{F}(X)$ of all hesitant fuzzy quasi-orders on X onto the partially ordered set of all complete $\mathcal{Q}$ -sublattices of $\mathcal{Q}_{\otimes}(X)$ satisfying conditions 2. and 3. of Theorem 6.2.*

Proof. According to the results of Theorem 6.2, we have to prove that for any two hesitant fuzzy quasi-orders F_1 and F_2 on X the following holds:

$$F_1 \subseteq F_2 \quad \Leftrightarrow \quad \langle w \circ F_2 = w \rangle \subseteq \langle w \circ F_1 = w \rangle. \quad (93)$$

Notice that the equivalence (93) ensures that φ is injection.

Let F_1 and F_2 be hesitant fuzzy quasi-orders on X . If $F_1 \subseteq F_2$ and $g \in \langle w \circ F_2 = w \rangle$, then we have:

$$g \subseteq g \circ F_1 \subseteq g \circ F_2 = g,$$

and, consequently, $g \circ F_1 = g$, i.e. $g \in \langle w \circ F_1 = w \rangle$. Conversely, assume that $\langle w \circ F_2 = w \rangle \subseteq \langle w \circ F_1 = w \rangle$. In that case, for each $x \in X$ we have that $xF_2 \in \langle w \circ F_2 = w \rangle \subseteq \langle w \circ F_1 = w \rangle$, i.e. $xF_2 \circ F_1 = xF_2$. Therefore, for arbitrary $x, y \in X$ we have

$$F_2(x, y) = xF_2(y) = (xF_2 \circ F_1)(y) = \bigcup_{z \in X} xF_2(z) \otimes F_1(z, y) \supseteq xF_2(x) \otimes F_1(x, y) \supseteq F_1(x, y),$$

and, hence, $F_1 \subseteq F_2$. From the above, we immediately obtain that (93) is satisfied. $\square$

In the sequel, we present the second characterization theorem and outline the corresponding order isomorphism analogously to Theorems 6.2 and 6.3. The proofs of the following theorems are analogous to those already presented and are therefore omitted.

Theorem 6.4. *Let $\mathcal{G}$ be a collection of hesitant fuzzy sets on X . Then there exists a hesitant fuzzy quasi-order F on X such that $\mathcal{G} = \langle F \circ w = w \rangle$ if and only if the following holds:*

- 1' $\mathcal{G}$ is a complete $\mathcal{Q}$ -sublattice of $\mathcal{Q}_{\otimes}(X)$;
- 2' every element of $\mathcal{G}$ can be expressed as a linear combination of principal elements of $\mathcal{G}$;
- 3' principal elements of $\mathcal{G}$ satisfy the reverse inclusion property.

Theorem 6.5. *The mapping $F \mapsto \langle F \circ w = w \rangle$ is a dual order isomorphism of the lattice $\mathcal{F}(X)$ of all hesitant fuzzy quasi-orders on X onto the partially ordered set of all complete $\mathcal{Q}$ -sublattices of $\mathcal{Q}_{\otimes}(X)$ satisfying conditions 2' and 3' of Theorem 6.4.*

7 Conclusion

In this paper, we studied hesitant fuzzy relation inequalities and equations defined over a power set quantale. We developed algorithms for computing the greatest solutions to systems of hesitant fuzzy relation inequalities and equations, contained in a given hesitant fuzzy set, including an algorithm for the greatest solution to the single equation of the form $w \circ R = w$. Furthermore, we provided algebraic characterizations of sets of solutions in terms of closure systems and established dual order isomorphisms between the lattice of hesitant fuzzy quasi-orders and lattices of sets of solutions. These results extend existing characterizations for fuzzy relations and provide a unified algebraic framework for analyzing hesitant fuzzy relation inequalities and equations.

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